Mesh-Based Piecewise Affine Abstraction with Polytopic Partitions for Nonlinear Systems

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***Abstract*—This paper considers the problem of piecewise affine abstraction with polytopic partitions of nonlinear systems, i.e., the over-approximation of nonlinear dynamics by a pair of piecewise affine functions over polytopic subdomains/partitions in the sense of the inclusion of all possible trajectories. Specifically, to tackle the "boundary effect" that may make the over-approximation incorrect for polytopic partitions, we propose two mesh-based affine abstraction approaches based on expanding the partitions to simultaneously find the polytopic partitions and the pair of piecewise functions over the partitions. The effectiveness of the proposed approaches are compared with existing methods using hyperrectangular partitions, and demonstrated by computing abstractions of swarm dynamics and applying them for swarm intent identification.**

***Index Terms*—Model/Controller reduction, Computational methods, Model validation**

I. Introduction

In recent years, various abstraction-based approaches have been proposed to analyze and control complex (nonlinear or hybrid) systems. The idea behind abstraction is to compute a simpler system which *over-approximates* the vector fields of the complex system dynamics (by allowing more system behaviors) such that existing analysis or synthesis tools can be leveraged for the simpler systems [1]–[3]. Importantly, the abstraction process is computed in a manner that includes all possible behaviors of the original system to preserve certain system properties of interest, e.g., robust reachability.

Literature Review. In a nutshell, abstraction is a systematic approximation method that partitions the vector field of a complex system into finite subregions, and then over-approximates its dynamics $f(\cdot)$ by a simpler inclusion model with $\overline{f}(\cdot)$ and $\underline{f}(\cdot)$ as bracketing functions or framers, such that for all x in each bounded subregion, $\underline{f}(x) \leq f(x) \leq \overline{f}(x)$, resulting in a hybrid system [4], [5]. Various abstraction methods have been proposed for linear systems [6], nonlinear systems [7], [8], hybrid systems [9], and uncertain systems [10], [11], as well as data-driven approaches when the model is unknown [12], [13]. In particular, two important classes of abstraction methods are *symbolic* approaches, e.g., [4], [7], and *hybridization*, e.g., [2], [14], [15].

Of specific interest to this paper is the hybridization method [2], [15], where the nonlinear vector fields are over-approximated with piecewise affine systems and the approximation error is accounted for with an additive disturbance. Building upon this framework, the authors in [8]–[11] proposed mesh-based methods to find a pair of piecewise affine functions/hyperplanes to bracket/frame the vector fields of the system dynamics and showed that the approximation errors (i.e., the distance between the hyperplanes) generally were lower than [2], [14], [15]. These abstractions were also shown to be effective for reachability analysis [15], control synthesis [9] and model discrimination [3], [11].

In these existing mesh-based approaches, the abstraction procedure is carried out over hyperrectangular partitions, either evenly-spaced [8]–[11] or based on the variation of the vector field inspired by Lebesgue integrals [14]. However, this restriction to hyperrectangular partitions may require a large number of partitions to achieve a desired abstraction accuracy. By contrast, this paper will consider more general polytopic partitions, similar in spirit to the data-driven method in [12] that is not directly applicable to our setting.

Contributions. In this paper, we propose mesh-based methods to compute piecewise affine abstractions for nonlinear systems with polytopic partitions, where the polytopic partitioning of the state space (as opposed to *a priori* chosen hyperrectangular partitions) and the bracketing piecewise hyperplanes over the original nonlinear dynamics are simultaneously obtained. Specifically, given that a uniform mesh may not be aligned with the polytopic subregions, we propose two approaches that rely on expanding the subregions to tackle the "boundary effect" such that the resulting piecewise affine abstraction is guaranteed to over-approximate the nonlinear dynamics over the entire domain. The first approach directly incorporates the region expansion into the abstraction algorithm and can be solved as a mixed-integer linear program, whereas the second consists of two successive steps to first find the piecewise affine abstraction with polytopic subregions over only the discrete set of mesh/grid points, before extending the over-approximation/abstraction to the entire continuous domain using linear programs with expanded subregions. The second approach is computationally faster than the first, while the first method often returns abstractions with smaller approximation errors.

Furthermore, it is noteworthy that our approaches form a building block to reduce abstraction errors that can be used within existing hybridization frameworks, e.g., [2], [9], [14], [15], including incremental and on-the-fly approaches, e.g., [1], [16], [17]; thus, the performance of these tools for reachability analysis and control synthesis can be improved.

Notation. For vectors $v, w \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{p\times q}$, $\|v\|_i$ and $\|M\|_i$ denote (induced) i-norm with $i \in \{1, 2, \infty\}$. $|M|$ is the element-wise absolute value matrix, while $v \leq$

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w and $v \odot w$ are, respectively, element-wise inequality and product. Further, $[n] \triangleq \{1, \ldots, n\}$ and $\mathbb{1}$ is a vector of ones.

II. Modeling Framework and Problem Statement

Consider a nonlinear dynamic system model $\mathcal{G}$:

$$x^+ \quad = f(x, u, w), \tag{1}$$

where $x \in \mathcal{X}$ is the system state at the current time instant with a closed interval domain $\mathcal{X} = [\underline{x}, \overline{x}]^n \subset \mathbb{R}^n$, $u \in \mathcal{U}$ is the control input with a closed interval domain $\mathcal{U} = [\underline{u}, \overline{u}]^m \subset \mathbb{R}^m$, $w \in \mathcal{W}$ is the disturbance/process noise with a closed interval domain $\mathcal{W} = [\underline{w}, \overline{w}]^{n_w} \subset \mathbb{R}^{n_w}$, and x^+ is $\dot{x}$ for continuous-time systems and the state at the next time step for discrete-time systems, while $f : \mathcal{X} \times \mathcal{U} \times \mathcal{W} \to \mathbb{R}^n$ is the vector field describing the nonlinear dynamics of the system, which belongs to several smoothness classes, for example, Lipschitz continuous (with Lipschitz constant λ), C^0, C^1, and C^2. To simplify the notation, we write $s = [x^\top \ u^\top \ w^\top]^\top$ as a stacked state-input-disturbance vector with the interval domain $\mathcal{S} = [\underline{s}, \overline{s}] = \mathcal{X} \times \mathcal{U} \times \mathcal{W}$.

Definition 1 (Polytopic Partition)**.** *A polytopic partition $\mathcal{P}$ of the closed and bounded region $\mathcal{S} \subset \mathbb{R}^{n+m+n_w}$ is a collection of p closed and bounded subregions:*

$$P_i \coloneqq \{s \mid G^i s \leq g^i\}, \tag{2}$$

for all $i \in [p]$ such that $\mathcal{S} \subseteq \bigcup_{i=1}^{p} P_i$ and $P_i \cap P_j = \partial P_i \cap \partial P_j$, $\forall i \neq j \in [p]$, where ∂P_i is the boundary of set P_i, G^i and g^i are real matrices/vectors.

Definition 2 (Uniform Mesh)**.** *A uniform mesh of $\mathcal{S}$ is a collection of evenly-spaced interval subregions $\mathcal{I}_i = [\underline{s}_i, \overline{s}_i] \subseteq \mathcal{S}$, called mesh elements, and its set of grid points $\mathcal{M}$ (i.e., vertices of $\mathcal{I}_i$) is defined by*

$$\{s | s = \underline{s}_i + K_i \odot \Delta_{s,i}, K_i \in \{0, 1, 2, \ldots, r_i - 1\}^{n+m+n_w}\},$$

where r_i is the resolution *or number of grid points along each dimension in each subregion $\mathcal{I}_i$ (or r only, if r_i is the same for all subregions), $\Delta_s \triangleq \frac{d_{s,i}}{r_i - 1}$ and $d_{s,i} \triangleq \overline{s}_i - \underline{s}_i$. Further, given $\tilde{\mathcal{M}} \subseteq \mathcal{M}$, we denote by $\mathrm{Int}(\tilde{\mathcal{M}} \cap \mathcal{I}_i)$ the union of all simplicial regions formed by the grid points of $\tilde{\mathcal{M}} \cap \mathcal{I}_i$.*

Definition 3 (Diameter [8])**.** *The diameter δ of a polytopic subregion is the greatest distance between its vertices.*

For each subregion $P_i \in \mathcal{P}$ as defined in Definition 1 that partitions the domain of interest, we aim to over-approximate/abstract the nonlinear f by a pair of affine functions $\underline{f}_i$ and $\overline{f}_i$ such that for all $s \in P_i$, the function $f(s)$ is bracketed/framed by the pair of affine functions, i.e., $\underline{f}_i(s) \leq f(s) \leq \overline{f}_i(s)$. These affine functions (i.e., framers) with respect to f over $P_i \in \mathcal{P}$ are chosen as

$$\underline{f}_i(s) = \underline{A}_i s + \underline{h}_i, \overline{f}_i(s) = \overline{A}_i s + \overline{h}_i, \tag{3}$$

where the matrices $\underline{A}_i$, $\overline{A}_i$, and the vectors $\underline{h}_i$ and $\overline{h}_i$ are constant and of appropriate dimensions. Let $(\underline{\mathcal{F}}, \overline{\mathcal{F}})$ be a pair of families of affine functions with $\underline{\mathcal{F}} = \{\underline{f}_1, \ldots, \underline{f}_p\}$ and $\overline{\mathcal{F}} = \{\overline{f}_1, \ldots, \overline{f}_p\}$. Then, the function $f : \mathcal{S} \to \mathbb{R}^n$ is over-approximated with a pair of affine families $(\underline{\mathcal{F}}, \overline{\mathcal{F}})$ over a partition $\mathcal{P}$ by the piecewise affine inclusion model $\mathcal{H}$:

$$\underline{A}_i s + \underline{h}_i \leq x^+ \leq \overline{A}_i s + \overline{h}_i, \ \forall i \in [p], s \in P_i. \tag{4}$$

Remark 1. *By construction, any two partitions P_i, $P_j \in \mathcal{P}$ may only overlap at their shared boundary (cf. Definition 1). Moreover, by design, $\forall s \in \partial P_i \cap \partial P_j$, $\underline{f}_k(s) \leq f(s) \leq \overline{f}_k(s)$ for both $k \in \{i, j\}$. In this case, we will choose framers via:* $\max_{k \in \{i,j\}} \underline{f}_k(s) \leq f(s) \leq \min_{k \in \{i,j\}} \overline{f}_k(s), \forall s \in \partial P_i \cap \partial P_j$.

To quantify the *quality* of our abstraction approaches, we utilize the following definition of their approximation errors.

Definition 4 (Approximation Error [8])**.** *Consider a polytopic partition $\mathcal{P} = \{P_i | i \in [p]\}$ of $\mathcal{S} \subset \mathbb{R}^{m+n+n_w}$. If a pair of affine families $(\underline{\mathcal{F}}, \overline{\mathcal{F}})$ over-approximates a nonlinear function f over the partition $\mathcal{P}$, then the approximation error with respect to the nonlinear dynamics is defined as $e(\underline{\mathcal{F}}, \overline{\mathcal{F}}) = \max_{i \in [p]} \max_{s \in P_i} \|\overline{f}_i(s) - \underline{f}_i(s)\|_\infty$.*

Further, we rely on the following results in our derivations.

Proposition 1 ([18, Theorem 4.1 & Lemma 4.3])**.** *Let S be an $(n + m + n_w)$-dimensional mesh element such that $S \subseteq \mathbb{R}^{n+m+n_w}$ with diameter δ (cf. Definition 3). Let $f : S \to \mathbb{R}$ be a nonlinear function and let f_l be the linear interpolation of $f(\cdot)$ evaluated at the vertices of the mesh element S. Then, the approximation error bound σ defined as the maximum error between f and f_l on S, i.e., $\sigma = \max_{s \in S}(|f(s) - f_l(s)|)$, is upper-bounded by*

(i) $\sigma \leq 2 \max_{s \in S} \|f(s)\|_\infty$, if $f \in C^0$ on S,
(ii) $\sigma \leq \lambda \delta_s$, if f is Lipschitz continuous on S,
(iii) $\sigma \leq \delta_s \max_{s \in S} \|f'(s)\|_2$, if $f \in C^1$ on S,
(iv) $\sigma \leq \frac{1}{2} \delta_s^2 \max_{s \in S} \|f''(s)\|_2$, if $f \in C^2$ on S,

where C^0, C^1 and C^2 are sets of continuous, continuously differentiable and twice continuously differentiable functions respectively, λ is the Lipschitz constant, $f'(s)$ is the Jacobian of $f(s)$, $f''(s)$ is the Hessian of $f(s)$ and δ_s satisfies $\delta_s \leq \sqrt{\frac{n+m+n_w}{2(n+m+n_w+1)}} \delta \triangleq \delta_s^*$. Further, the smallest guaranteed approximation error σ^* is obtained for cases (i)–(iv) with δ_s^*.

We now formulate the problem of interest to this paper.

Problem 1 (Affine Abstraction with Polytopic Partitions)**.** *For a given nonlinear n-dimensional vector field $f(s)$ with $s \in \mathcal{S}$ and a given desired accuracy ε_f, simultaneously find a polytopic partition $\mathcal{P} = \{P_1, \ldots, P_p\}$ and a pair of n-dimensional families of affine functions $\overline{\mathcal{F}} = \{\overline{f}_1, \ldots, \overline{f}_p\}$ and $\underline{\mathcal{F}} = \{\underline{f}_1, \ldots, \underline{f}_p\}$ such that:*

$$e(\underline{\mathcal{F}}, \overline{\mathcal{F}}) \leq \varepsilon_f, \ \underline{f}_i(s) \leq f_i(s) \leq \overline{f}_i(s), \forall s \in P_i, \forall i \in [p], \tag{5}$$

where $e(\underline{\mathcal{F}}, \overline{\mathcal{F}})$ is the approximation error (cf. Definition 4). The pair of affine families $(\underline{\mathcal{F}}, \overline{\mathcal{F}})$ is then the abstracted piecewise affine inclusion model (i.e., piecewise affine abstraction of the nonlinear dynamics).

III. Main Result on Abstraction

In this section, optimization-based approaches are introduced for finding piecewise affine abstractions with polytopic partitions of the nonlinear system (1), i.e., for solving Problem 1. First, we consider the problem with a given number of subregions p (determined by a given number of hyperplanes L) in Section III-A. Then, in Section III-B, we discuss a recursive procedure to reduce computational complexity while searching for p to satisfy the desired accuracy ε_f.

A. Mesh-Based Piecewise Affine Abstraction Approach

It is generally nontrivial to abstract a nonlinear function $f(s)$ over its entire continuous domain. Therefore, similar to [8], [9], we adopt a mesh-based method to abstract the function at only the grid points and rely on the result on interpolation errors in Proposition 1 to compensate for the generalization error when extending it to the entire continuous domain. However, in contrast to previous approaches in [8], [9], instead of uniformly partitioning the domain of interest with hyperrectangles and performing the abstraction procedure with the grid points over the *a priori* determined hyperrectangles, we propose to partition the domain using polytopic regions that are computed simultaneously with the abstraction process, with the goal of improving the quality of the abstraction in terms of reducing its approximation error.

1) Polytopic Partitions: We consider $L \geq 1$ hyperplanes to partition the domain of interest, $\mathcal{S}$, into polytopic subregions, where each hyperplane $i \in [L]$ is represented by:

$$G_i^\top s = 1, \tag{6}$$

with to-be-determined $G_i \in \mathbb{R}^{n+m+n_w}$, and these hyperplanes divide the domain of interest into at most 2^L polyhedral sets intersecting only on the boundary (i.e., $p \leq 2^L$). Specifically, each polytopic subregion $l \in [2^L]$ is defined by

$$P_l \triangleq \{s \in \mathcal{S} \mid \beta_{l,i}(G_i^\top s) \leq \beta_{l,i}, \forall i \in [L]\}, \tag{7}$$

with (distinct) permutation vectors $\beta_l \in \{-1,1\}^{n+m+n_w}$.

2) Piecewise Affine Abstraction with Polytopic Regions: Armed with the above description of polytopic partitions, we provide a lemma that simultaneously finds the polytopic partitions and the affine abstractions for each subregion that only considers the grid points. This lemma is an extension of the result in [12], where a similar problem for data-driven piecewise affine fitting with polytopic partitions is considered, but the $\underline{A}_i$ and $\overline{A}_i$ matrices for the piecewise affine abstraction in (4) are now allowed to be different.

Lemma 1. *Given a nonlinear function $f : \mathcal{S} \to \mathbb{R}^n$ with a given closed and bounded region $\mathcal{S}$, and let $\mathcal{M} = \{s_1, s_2, \ldots, s_J\}$ be a set of J grid points in the region $\mathcal{S}$ (cf. Definition 2) and L be the desired number of hyperplanes that partitions the region. The piecewise affine hyperplanes*

$$\overline{f}_l(s) = \overline{A}_l\, s + \overline{h}_l, \underline{f}_l(s) = \underline{A}_l\, s + \underline{h}_l, \quad \text{if } G^l s \leq g^l,$$

for all $l \in [2^L]$, bound f from above and below at (only) the set of grid points, i.e., (4) holds for all $s \in \mathcal{M} \cap P_l$, where $\overline{A}_l, \underline{A}_l, \overline{h}_l$, $\underline{h}_l$, G^l, and g^l are obtained from the following mixed-integer linear program (MILP):

$$\begin{aligned}
&\min_{\theta,\overline{A}_l,\underline{A}_l,\overline{h}_l,\underline{h}_l,G_i,z'_{j,l},z"_{j,l,i}} \|\theta\|_q \\
\text{s.t. } & \overline{A}_l\, s_j + \overline{h}_l \geq f(s_j) + M(\mathbb{1} - z'_{j,l}), && \text{(8a)}\\
& \underline{A}_l\, s_j + \underline{h}_l \leq f(s_j) - M(\mathbb{1} - z'_{j,l}), && \text{(8b)}\\
& (\overline{A}_l - \underline{A}_l)x_j + \overline{h}_l - \underline{h}_l \leq \theta - M(\mathbb{1} - z'_{j,l}), && \text{(8c)}\\
& \beta_{l,i} G_i^\top s_j \leq \beta_{l,i} - M(v - z''_{j,l,i}), && \text{(8d)}\\
& \textstyle\sum_{i=1}^{L} z''_{j,l,i} - L + 1 \leq z'_{j,l}, && \text{(8e)}\\
& z'_{j,l}, z''_{j,l,i} \in \{0,1\}, \forall l \in [2^L], \forall j \in [J], \forall i \in [L], && \text{(8f)}
\end{aligned}$$

with a large enough M, a pre-defined/given vector norm q and β_l defined in (7), while G^l and g^l are row-wise concatenations of $\beta_{l,i} G_i^\top$ and $\beta_{l,i}$, respectively.

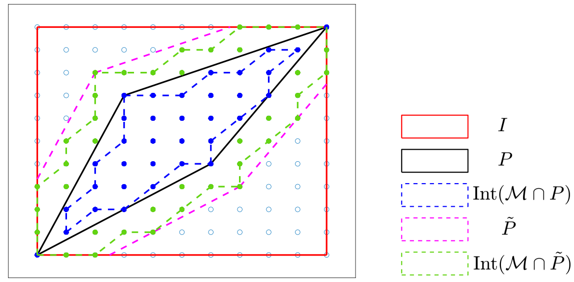


Fig. 1: Illustration of the region expansion strategy.

Proof. This result holds by construction. Each grid point v_j will be in one of the partitions described by $G^l s \leq g^l$, where $z''_{j,l,i} = 1$ for all $i \in [L]$. For this l, (8e) ensures that $z'_{j,l} = 1$, and in turn, (8a) and (8b) guarantee that the grid points v_j will be bounded by $f_{u,l}$ and $f_{b,l}$, while (8c) encode its approximation error θ. □

Remark 2. *Note that when $q = 1$, Lemma 1 can also be carried out dimension-wise for the vector-valued nonlinear function f, which results in smaller optimization problems.*

3) Mesh-Based Methods for Polytopic Regions: However, when sampling points in a polytopic region, one of the "complications" is that uniform meshes, as were considered in [8], [9], no longer conveniently align with the polytopic subregions, as shown in Figure 1 for a polytopic region described by black solid lines. Although it is theoretically possible to tailor a non-uniform mesh to coincide with the polytope, this procedure is non-trivial and can be computationally expensive, especially when the non-uniform mesh needs to be generated as a function of the yet to-be-determined polytopic regions, which is the case we consider.

Hence, a specific contribution of this paper is to find computationally efficient algorithms to generate appropriate meshes for polytopic regions, which may not necessarily provide the optimal non-uniform meshes. To achieve this for an arbitrary polytopic region P, we can first find the hyperrectangle I that covers/contains the polytope with minimum volume. Then, we generate a uniform mesh over I with the set of grid points $\mathcal{M}$, similar to previous approaches. This hyperrectangle that covers the polytope P is illustrated in red in Figure 1, while the grid points are marked in light blue. On one hand, since many of the grid points are not within the (black) polytope P, the computation of hyperplanes that frame the nonlinear function (i.e., affine abstraction) over all the grid points in $\mathcal{M}$ will result in unnecessarily suboptimal approximation errors. On the other hand, there exists a "boundary effect" such that the resulting abstraction will not over-approximate the nonlinear function over the entire the polytopic region if we only use the grid points within the polytope, i.e., the (dark) blue grid points given by $\mathcal{M} \cap P$. This is because our correction mechanism for generalizing the mesh-based method to the entire continuous polytopic domain using Proposition 1 only applies for the interpolation over simplices formed by the grid points in $\mathcal{M} \cap P$, resulting in an abstraction over the region enclosed by the blue dashed lines, Int$(\mathcal{M} \cap P)$, instead of P (black).

4) Region expansion method: In order to address this "boundary effect," we propose a region expansion method that seeks to find a larger polytopic region, $\tilde{P} \supseteq P$, that contains/covers the polytopic region P, such that the affine abstraction over the grid points in this enlarged polytopic region, i.e., Int$(\mathcal{M} \cap \tilde{P})$, is a true over-approximation of the nonlinear function f over the original polytope P.

Specifically, we propose to expand the polytopic region P by extending its boundaries outwards by the distance between adjacent grid points in each dimension such that the enlarged polytope $\tilde{P}$ is guaranteed to include the immediate neighbor grid points to the boundaries. This procedure can be formalized using the following lemmas.

Lemma 2. *Given a polytopic region $P = \{s \in \mathcal{S} \mid Gs \leq g\}$, and a tight hyperrectangle $I = [\underline{s}, \overline{s}] \supseteq P$ with width $d_s = \overline{s} - \underline{s}$, a uniform mesh over the hyperrectangle I with resolution r, the expanded polytopic region $\tilde{P}$ that result from expanding each element $s \in P$ by an interval $[-\Delta_s, \Delta_s]$ with $\Delta_s = \frac{d_s}{r-1} - \epsilon$ and a very small constant ϵ, is given by*

$$\tilde{P} = \{s \in \mathcal{S} \mid Gs \leq g + |G|\Delta_s\}. \quad (9)$$

Proof. By a helpful result in [19, Lemma 1], for each $s - \Delta_s \leq s' \leq s + \Delta_s$, we have $Gs' \geq Gs - |G|\Delta_s$. Hence, $Gs - |G|\Delta_s \leq Gs' \leq g$, which simplifies to (9). □

Lemma 3. *Given an interval region $\mathcal{S} = [\underline{s}, \overline{s}]$ and a uniform mesh with $\mathcal{M} = \{s | s = \underline{s} + K \odot \Delta_s, K \in \{0, 1, 2, \ldots, r-1\}^{n+m+n_w}\}$ with resolution r, where $\Delta_s = \frac{d_s}{r-1}$ and $d_s = \overline{s} - \underline{s}$, there exists $s_m \in \mathcal{M}$ for any $s \in \mathcal{S}$ such that*

$$s \in [s_m, s_m + \Delta_s] \subset [\max(s - \Delta_s, \underline{s}), \min(s + \Delta_s, \overline{s})]. \quad (10)$$

Proof. For an arbitrary $s \in \mathcal{S}$, let $s_m = \underline{s} + K \odot \Delta_s$ and

$$K_{(j)} = \begin{cases} 0, & s_{(j)} = \underline{s}_{(j)}, \\ \left\lceil \frac{s_{(j)} - \Delta_{s,(j)} - \underline{s}_{(j)}}{\Delta_{s,(j)}} \right\rceil, & \underline{s}_{(j)} < s_{(j)} \leq \overline{s}_{(j)}, \end{cases} \quad (11)$$

$\forall j \in [n + m + n_w]$, where the subscript (j) denotes the j-th element. We will show that (10) holds for both cases in (11).

Case I: $s_{(j)} = \underline{s}_{(j)} \Rightarrow s_{m,(j)} = \underline{s}_{(j)} \Rightarrow \max(s_{(j)} - \Delta_{s,(j)}, \underline{s}_{(j)}) = \underline{s}_{(j)} \leq s_{m,(j)} \leq s_{(j)} \leq s_{m,(j)} + \Delta_{s,(j)} \leq \min(s_{(j)} + \Delta_{s,(j)}, \overline{s}_{(j)}) = s_{(j)} + \Delta_{s,(j)}$, which simplifies to (10).

Case II: When $\underline{s}_{(j)} < s_{(j)} \leq \overline{s}_{(j)}$ and from definition, we have $\overline{s}_{(j)} = \underline{s}_{(j)} + (r-1)\Delta_{s,(j)}$, thus $\frac{\overline{s}_{(j)} - \Delta_{s,(j)} - \underline{s}_{(j)}}{\Delta_{s,(j)}} = r - 2 \geq \frac{s_{(j)} - \Delta_{s,(j)} - \underline{s}_{(j)}}{\Delta_{s,(j)}} > \frac{\underline{s}_{(j)} - \Delta_{s,(j)} - \underline{s}_{(j)}}{\Delta_{s,(j)}} = -1$. Hence, $r - 2 \geq K_{(j)} = \left\lceil \frac{s_{(j)} - \Delta_{s,(j)} - \underline{s}_{(j)}}{\Delta_{s,(j)}} \right\rceil \geq 0$. Consequently,

$$\begin{aligned} s_{m,(j)} &= \underline{s}_{(j)} + K_{(j)}\Delta_{s,(j)} \geq \underline{s}_{(j)} \\ s_{m,(j)} + \Delta_{s,(j)} &= \underline{s}_{(j)} + K_{(j)}\Delta_{s,(j)} + \Delta_{s,(j)} \\ &\leq \underline{s}_{(j)} + (r-2)\Delta_{s,(j)} + \Delta_{s,(j)} = \overline{s}_{(j)}. \end{aligned} \quad (12)$$

From the definition of the ceiling function, we have

$$\frac{s_{(j)} - \Delta_{s,(j)} - \underline{s}_{(j)}}{\Delta_{s,(j)}} \leq \left\lceil \frac{s_{(j)} - \Delta_{s,(j)} - \underline{s}_{(j)}}{\Delta_{s,(j)}} \right\rceil \leq \frac{s_{(j)} - \Delta_{s,(j)} - \underline{s}_{(j)}}{\Delta_{s,(j)}} + 1,$$

$$\Rightarrow s_{m,(j)} + K_{(j)}\Delta_{s,(j)} \leq \underline{s}_{(j)} + \left(\frac{s_{(j)} - \Delta_{s,(j)} - \underline{s}_{(j)}}{\Delta_{s,(j)}} + 1\right)\Delta_{s,(j)} = s_{(j)},$$

$$s_{m,(j)} \geq \underline{s}_{(j)} + \frac{s_{(j)} - \Delta_{s,(j)} - \underline{s}_{(j)}}{\Delta_{s,(j)}}\Delta_{s,(j)} = s_{(j)} - \Delta_{s,(j)},$$

$$\Rightarrow s_{(j)} - \Delta_{s,(j)} \leq s_{m,(j)} \leq s_{(j)} \leq s_{m,(j)} + \Delta_{s,(j)} \leq s_{(j)} + \Delta_{s,(j)}.$$

Combining with the results of $s_{m,(j)} \geq \underline{s}_{(j)}$ and $s_{m,(j)} + \Delta_{s,(j)} \leq \overline{s}_{(j)}$ in (12), we obtain (10). □

Intuitively, by Lemma 2, we ensure that $[s - \Delta_s, s + \Delta_s]$ is a subset of $\tilde{P}$ for all $s \in P$ (depicted in magenta in Figure 1), and by Lemma 3, we show that there exists a mesh element $[s_m, s_m + \Delta_s] \subset I$ that contains s and is contained in $[s - \Delta_s, s + \Delta_s]$. In other words, each point $s \in P$ is contained in a mesh element of $\tilde{P}$, hence the correction mechanism in Proposition 1 that applies to simplices formed by the grid points in $\mathcal{M} \cap \tilde{P}$, i.e., Int$(\mathcal{M} \cap \tilde{P})$ (depicted by green dashed lines in Figure 1), is guaranteed to contain P, i.e., Int$(\mathcal{M} \cap \tilde{P}) \supset P$, and thus, an affine abstraction with domain Int$(\mathcal{M} \cap \tilde{P})$ also applies to the (black) region P.

Next, we provide the necessary corrections to account for interpolation errors and "boundary effects" using two region expansion methods. As mentioned above, we will show that the "boundary effects" can be overcome by expanding each polytopic region using Lemmas 2 and 3, while interpolation error corrections can be done via Proposition 1 with the expanded/enlarged polytopic regions.

a) Method I: The first method directly combines the result from Lemma 2 into the MILP in (8). Instead of only considering the grid points in partitions P_l, we consider those in the enlarged polytopic regions $\tilde{P}_l$, enabling us to simultaneously find a polytopic partition $P = \{P_l\}_{l \in [2^L]}$ and a pair of piecewise affine hyperplanes that over-approximates the nonlinear function f over the entire domain $\mathcal{S}$.

Theorem 1. *Given a nonlinear function $f : \mathcal{S} \to \mathbb{R}^n$ with a given closed and bounded region $\mathcal{S}$, and let $\mathcal{M} = \{s_1, s_2, \ldots, s_J\}$ be a set of J grid points in region $\mathcal{S}$ and L be the desired number of hyperplanes that partitions the region. The piecewise affine hyperplanes*

$$\overline{f}_l(s) = \overline{A}_l s + \overline{h}_l + \sigma_I, \underline{f}_l(s) = \underline{A}_l s + \underline{h}_l - \sigma_I, \text{ if } G^l s \leq g^l, \quad (13)$$

for all $l \in [2^L]$, over-approximate/abstract f over the entire region $\mathcal{S}$, where $\overline{A}_l, \underline{A}_l, \overline{h}_l, \underline{h}_l, G^l$ and g^l are obtained from the MILP in (8) *with* (8d) *replaced by*

$$\beta_{l,i} G_i^\top s_j \leq \beta_{l,i} + |G_i|^\top \Delta_s - M(1 - z''_{j,l,i}), \quad (14)$$

σ_I is the interpolation error σ^* from Proposition 1, and the element-wise absolute value $|G_i|$ can be encoded with mixed-integer constraints corresponding to if-else *statements.*

Proof. This result holds by construction. We know from Lemma 2 that $[s - \Delta_s, s + \Delta_s] \subset \tilde{P}_l$ for any $s \in P_l$, where $P_l = \{G^l s \leq g^l\}$ and $\tilde{P}_l = \{G^l s \leq g^l + \Delta_l\}$. According to Lemma 3, we can always find an interval region such that $s \in [s_m, s_m + \Delta_s] \subset [\max(s - \Delta_s, \underline{s}), \min(s + \Delta_s, \overline{s})]$, where $s_m \in \mathcal{M}$, $\underline{s}$ and $\overline{s}$ are upper and lower bounds of the uniform mesh as defined in Definition 2. From $[s_m, s_m + \Delta_s] \subset [\max(s - \Delta_s, \underline{s}), \min(s + \Delta_s, \overline{s})]$, we have $[s_m, s_m + \Delta_s] \subset$ Int$(\mathcal{M})$ and $[s_m, s_m + \Delta_s] \subset \tilde{P}_l$. Therefore, $\forall s \in P_l, s \in$ Int$(\mathcal{M} \cap \tilde{P}_l)$. Consequently, $P_l \subset$ Int$(\mathcal{M} \cap \tilde{P}_l)$. According to [8, Lemma 2], if $\overline{A}_l s + \overline{h}_l \geq f(s) \geq \underline{A}_l s + \underline{h}_l$ for all $s \in \mathcal{M}$, then $\overline{f}_l(s) \geq f(s) \geq \underline{f}_l(s)$ for all $s \in$ Int$(\mathcal{M} \cap \tilde{P}_l)$. Therefore, $\overline{f}_l$ and $\underline{f}_l$ abstract/frame f over P_l. □

Note, however, that the integer constraints due to $|G_i|$ in (14) (encoded with if-else statements) typically result in increased branches/cuts in MILP solvers and in turn, may increase the computation time to find the polytopic partitions.

b) Method II: To alleviate the issue of computation time with Method I, we propose a two-step technique, where the first step over-approximates the function f over only the grid points using Lemma 1 to obtain the polytopic partitions and the second step then finds the affine hyperplanes over each enlarged polytopic partition $\tilde{P}_l$, independently.

Theorem 2. *Suppose Theorem 1 hold. Then, the piecewise affine hyperplanes defined in* (13) *for all* $l \in [2^L]$, *over-approximate* f *over the entire region* $\mathcal{S}$, *where* $\overline{A}_l$, $\underline{A}_l$, $\overline{h}_l$ *and* $\underline{h}_l$ *are obtained from the following linear program (LP):*

$$\min_{\theta,\overline{A}_l,\underline{A}_l,\overline{h}_l,\underline{h}_l} \|\theta\|_q$$
$$\text{s.t. } \overline{A}_l\, s_j + \overline{h}_l \geq f(s_j), \underline{A}_l\, s_j + \underline{h}_l \leq f(s_j), \tag{15a}$$
$$(\overline{A}_l - \underline{A}_l)s_j + \overline{h}_l - \underline{h}_l \leq \theta, \tag{15b}$$
$$\forall\, s_j \in \{s \in \mathcal{V} \mid G^l s \leq g^l + |G^l|\Delta_s\}, \tag{15c}$$

while G^l *and* g^l *are obtained from the MILP in* (8), *with* σ_I *as the interpolation error* σ^* *from Proposition 1.*

Proof. This result holds by a similar reasoning as Theorem 1. Constraint (15) ensures that each grid point in the enlarged polytopic partition $\tilde{P}_l$ will be bounded by $f_{u,l}$ and $f_{b,l}$ and by [8, Lemma 2], the interpolation error σ_I enables us to extend the abstraction over the grid points of the expanded/enlarged polytopic regions to the entire continuous region. □

B. Recursive Approach for Piecewise Affine Abstraction

Theoretically, we can solve Problem 1 by applying either Theorem 1 or Theorem 2 with increasing L and/or the number of grid points/resolution r_i to achieve any desired accuracy ε_f. However, the increase of L or r_i may lead to increasing integer constraints or to very large optimization problems, where computational time and memory can become an issue for systems with large dimensions.

Thus, we provide an approach that can be recursively implemented with a smaller L, e.g., $L = 1$ and with less grid points/resolution in each recursion. In this procedure, we begin with the abstraction of the entire domain with polytopic regions with a (chosen) small L and resolution using Theorem 1 or 2. If the desired accuracy is not met for any of the polytopic regions, we will further partition that polytopic partition by re-applying Theorem 1 or 2 with the same small L and resolution but for a smaller polytopic region to obtain a better desired accuracy. This procedure can be repeated as needed until the desired accuracy is achieved. The algorithm for this procedure (omitted for brevity) is similar to [8, Algorithm 1], except that its theorem is replaced by our Theorem 1 or 2 and that the partitions are found with our theorems instead of being manually constructed.

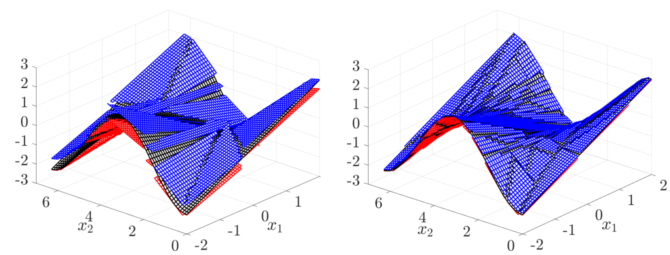
IV. Simulation Results

In this section, all simulations are implemented in MATLAB on a 2.2 GHz Intel Core i7 CPU with 16 GB RAM.

A. Dubins Car Dynamics ($f(\mathbf{x}) = x_1 cos(x_2)$)

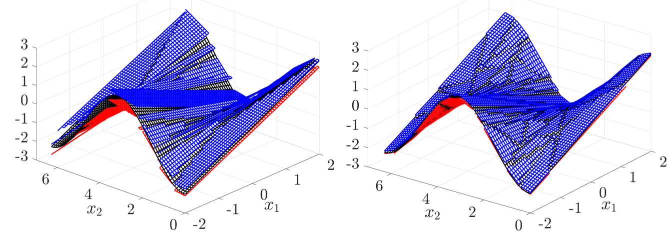
We first recursively applied Theorem 1 and 2 with $L = 1$ as described in Section III-B to over-approximate one of the vector fields of a Dubins car dynamics, $f(\mathbf{x}) = x_1 cos(x_2)$ over the interval $[-2, 2] \times [0, 2\pi]$. Similar results are obtained for the rest of the Dubins car model but are omitted for brevity. As shown in Figure 2 and 3, both approaches can obtain tight abstractions for two different desired accuracies. We further compared the performance of the proposed recursive affine abstraction approaches with those in [8] and [9] for the piecewise affine abstraction of the same function over the same domain for three different desired abstraction accuracies, i.e., $\epsilon_f \in \{0.05, 0.1, 0.2\}$. The simulation results are summarized in Table I, which shows that our first method (Theorem 1) needs the smallest number of partitions for all desired accuracies, but it takes longer computation times than Method II (Theorem 2) and the existing approaches in [8], [9], since it involves more integer constraints.

In addition, when the number of partitions is set to be the same, Figure 4 shows that the proposed abstraction approaches achieve a better accuracy than the abstraction approach in [8] (Similarly, Method II performs better but its figure is omitted due to space limitation). This is likely because the approach in [8] partitions the domain evenly with hyperrectangles, while the proposed methods simultaneously determine the polytopic partitions along with the abstraction.



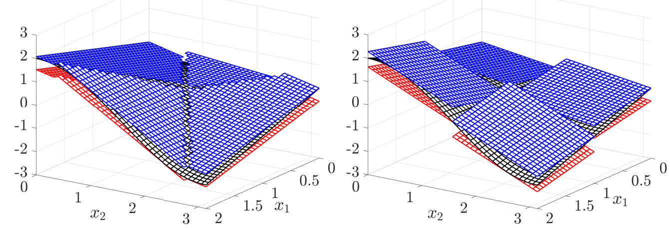
(a) $\epsilon_f = 0.8$ (b) $\epsilon_f = 0.05$

Fig. 2: Piecewise affine abstraction of $x_1 cos(x_2)$ using Method I (Theorem 1) with different desired accuracies.



(a) $\epsilon_f = 0.8$ (b) $\epsilon_f = 0.05$

Fig. 3: Piecewise affine abstraction of $x_1 cos(x_2)$ using Method II (Theorem 2) with different desired accuracies.



(a) Abstraction error $e = 0.5605$ (b) Abstraction error $e = 0.6596$

Fig. 4: Comparison of Method I (Theorem 1) with polytopic partitions (**left**) and the method in [8] with hyperectangular partitions (**right**), with the number of partitions fixed at 4.

TABLE I: Results of piecewise affine abstraction of nonlinear function $x_1 \cos x_2$ for varying desired accuracies ε_f.

	Desired Accuracy, ε_f	0.2	0.1	0.05
(i) C^2 function (Theorem 1)	No. of Subregions	58	108	210
	CPU Time (s)	2264	4524	8653
(ii) C^2 function (Theorem 2)	No. of Subregions	61	121	257
	CPU Time (s)	391.3	1003	2888
[8] C^2 function	No. of Subregions	64	232	256
	CPU Time (s)	19.68	69.32	86.34
[9] Lipschitz function	No. of Subregions	256	1024	4096
	Comp. Time (s)	57.18	214.40	786.88

TABLE II: Abstraction results using two partition hyperplanes for nonlinear function $x_1 \cos x_2$ over $[0, 2] \times [0, \pi]$.

Method	Theorem 1	Theorem 2	Data-Driven [12]
CPU Time (s)	2765	53.73	142.10
Approximation Error (e)	0.5605	0.5757	6.8005

TABLE III: Maximum number of steps and mean CPU time for model discrimination when using abstractions of swarm dynamics based on [11] and the proposed methods.

	Model	I	II	III
Method I (Theorem 1) with Polytopic Partitions	Max. No. of Steps	4	8	15
	Mean CPU Time (s)	1.81	2.81	4.08
Method II (Theorem 2) with Polytopic Partitions	Max. No. of Steps	4	8	18
	Mean CPU Time (s)	1.81	2.80	4.64
Approach in [11] with Hyperrectangular Partitions	Max. No. of Steps	4	9	37
	Mean CPU Time (s)	1.89	2.82	8.68

Moreover, we compared our abstraction approaches with a adapted version of [12] that also uses an optimization-based method to obtain polytopic partitions. The same nonlinear function on the domain $[0, 2] \times [0, \pi]$ is considered and we only partitioned the domain with two partition hyperplanes (i.e., $L = 2$) for simplicity. As shown in Table II, our first method (Theorem 1) has the best performance in terms of the abstraction error, while Method II (Theorem 2) has the smallest CPU time with a slightly increased abstraction error.

B. Application to Model Discrimination of Swarm Intents

Next, we compare the proposed Methods I and II (Theorems 1 and 2) with the approach using hyperrectangular partitions in [11] in terms of their effectiveness for the model discrimination problem. In particular, we consider the swarm intent identification example in [11] with three swarm intent models (see [11] for details): the swarm intends to move towards the centroid of the swarm (Model I), the swarm moves away from the centroid (Model II) and the swarm agents do not interact with each other (Model III), as well as 10 sampled input-output trajectories of 40 time steps each with the same inputs, initial conditions and noise sequences.

To investigate the effects of using different abstraction methods on the number of time steps needed for model discrimination, the number of partitions is set to be equal to 4 for all methods. As shown in Table III, when using abstraction models based on Theorems 1 and 2, the model discrimination algorithm in [11] takes fewer steps to discriminate Models II and III than when using abstractions obtained from the approach in [11]. Further, the mean computation time for the model discrimination algorithm is also reduced when using abstractions based on our proposed approaches, presumably due to the improved model over-approximations.

V. Conclusion

In this paper, we proposed mesh-based approaches for piecewise affine abstraction of nonlinear systems with polytopic partitions. To tackle the boundary effect that may make the abstraction incorrect, two optimization-based abstraction approaches are developed to obtain the polytopic partitions and affine abstractions simultaneously. Hence, the original dynamics can be over-approximated by a pair of piecewise affine functions with polytopic regions. Specifically, the first approach directly incorporates a region expansion strategy, while the second compensates for the boundary effect and interpolation errors in a 2-step procedure. Finally, we demonstrated the effectiveness of our approaches and illustrated their application to model discrimination via simulations.

References

[1] M. Althoff, O. Stursberg, and M. Buss, "Reachability analysis of nonlinear systems with uncertain parameters using conservative linearization," in *IEEE Conf. on Decision and Control*, 2008, pp. 4042–4048.

[2] A. Girard and S. Martin, "Synthesis for constrained nonlinear systems using hybridization and robust controller on symplices," *IEEE Trans. on Automatic Control*, vol. 57, no. 4, pp. 1046–1051, 2012.

[3] K. Singh, Y. Ding, N. Ozay, and S. Z. Yong, "Input design for nonlinear model discrimination via affine abstraction," in *IFAC PaperOnLine*, vol. 51, no. 16, 2018, pp. 175–880.

[4] P. Tabuada, *Verification and Control of Hybrid Systems: A Symbolic Approach*. Springer, 2009.

[5] E. Asarin, T. Dang, and A. Girard, "Hybridization methods for the analysis of nonlinear systems," *Acta Informatica*, vol. 43, no. 7, pp. 451–476, 2007.

[6] A. Girard, "Approximately bisimilar finite abstractions of stable linear systems," in *ACM International Conference on Hybrid Systems: Computation and Control*. Springer, 2007, pp. 231–244.

[7] G. Reissig, "Computing abstractions of nonlinear systems," *IEEE Trans. on Automatic Control*, vol. 56, no. 11, pp. 2583–2598, 2011.

[8] K. Singh, Q. Shen, and S. Z. Yong, "Mesh-based affine abstraction of nonlinear systems with tighter bounds," in *IEEE Conference on Decision and Control*, 2018, pp. 3056–3061.

[9] V. Alimguzhin, F. Mari, I. Melatti, I. Salvo, and E. Tronci, "Linearizing discrete-time hybrid systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 10, pp. 5357–5364, 2017.

[10] Q. Shen and S. Z. Yong, "Robust optimization-based affine abstractions for uncertain affine dynamics," in *American Control Conference*, July 2019, pp. 2452–2457.

[11] Z. Jin, Q. Shen, and S. Z. Yong, "Optimization-based approaches for affine abstraction and model discrimination of uncertain nonlinear systems," in *IEEE Conference on Decision and Control*, 2019.

[12] S. Sadraddini and C. Belta, "Formal guarantees in data-driven model identification and control synthesis," in *International Conference on Hybrid Systems: Computation and Control*. ACM, 2018, pp. 147–156.

[13] Z. Jin, Q. Shen, and S. Z. Yong, "Data-driven model invalidation for unknown lipschitz continuous systems via abstraction," in *IEEE Conference on Decision and Control*, 2020, to appear.

[14] S.-i. Azuma, J.-i. Imura, and T. Sugie, "Lebesgue piecewise affine approximation of nonlinear systems," *Nonlinear Analysis: Hybrid Systems*, vol. 4, no. 1, pp. 92–102, 2010.

[15] T. Dang, O. Maler, and R. Testylier, "Accurate hybridization of nonlinear systems," in *ACM International Conference on Hybrid Systems: Computation and Control*, 2010, pp. 11–20.

[16] Z. Han and B. H. Krogh, "Reachability analysis of nonlinear systems using trajectory piecewise linearized models," in *the American Control Conference*, 2006, pp. 1505–1510.

[17] S. M. Hassaan, M. Khajenejad, S. Jensen, Q. Shen, and S. Z. Yong, "Incremental affine abstraction of nonlinear systems," *IEEE Control Systems Letters*, vol. 5, no. 2, pp. 653–658, 2021.

[18] M. Stämpfle, "Optimal estimates for the linear interpolation error for simplices," *Jour. of Approximation Theory*, vol. 103, pp. 78–90, 2000.

[19] D. Efimov, T. Raïssi, S. Chebotarev, and A. Zolghadri, "Interval state observer for nonlinear time varying systems," *Automatica*, vol. 49, no. 1, pp. 200–205, 2013.