# Fault-Tolerant Attitude Tracking Control for a Quadrotor Aircraft

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***Abstract*— A quaternion-based attitude tracking scheme for a quadrotor subject to external disturbances and partial loss of rotor effectiveness faults is proposed in this paper. In attitude tracking controller design, the indirect robust adaptive control technique is employed to compensate the external disturbances and gyroscopic term, and stable attitude trajectory tracking is achieved. Then, the desired attitude tracking torques are maintained by designing the motor torques in the case of partial loss of rotor effectiveness faults. Based on the on-line estimation of the rotor effectiveness factor, an adaptive fault-tolerant controller is proposed to generate motor torques, where the rotor faults are handled without using the fault detection and diagnosis (FDD) mechanism. Simulation results are provided to demonstrate the performance of the proposed scheme against rotor faults.**

## I. Introduction

Unmanned four-rotor helicopters (quadrotors) have recently attracted an increasing interest due to their vertical landing/taking-off capability, great maneuverability, and low cost. The attitude tracking controller of the quadrotor is an important feature since it allows the vehicle to track a desired orientation and hence, prevents the vehicle from flipping over and crashing when the quadrotor performs the desired maneuvers [1]. Although several control methods have been used to achieve the stability of quadrotors in the literature, design of nonlinear robust attitude tracking controllers for quadrotors in the presence of external disturbances and actuator faults remains a challenging task [2], [3].

Several attitude control approaches have been developed in the literature for quadrotor aircraft in fault-free case. For example, a quaternion-based PD$^2$ feedback control scheme was proposed to achieve exponential attitude stabilization of a rigid quadrotor vehicle [1]. In [4], dynamic inversion and feedback linearization technique were used to design a station-keeping and tracking controller for a quadrotor. Despite the attitude of the quadrotor aircraft can be controlled in the aforementioned literature, lacking of robustness is a common defect. The paper [5] developed a nested-saturation-based nonlinear controller to increase robustness for the stabilization of a rotary-wing aircraft, where Lyapunov analysis was used to establish the convergence property for the nonlinear model of the quadrotor. By using an extended observer to estimate a class of time-varying disturbance, a sliding mode controller is designed to stabilize the attitude of a quadrotor in the presence of external disturbances in [6]. In [7], a robust adaptive attitude tracking scheme was proposed to eliminate the parametric and nonparametric uncertainties of a quadrotor unmanned aerial vehicle.

While the above control schemes assume that there is no occurrence of actuator faults, only a few papers in the literature are available within the literature dealing with actuator faults for the quadrotor. Since quadrotor systems are dynamically unstable and hard to control [8], a minor unexpected actuator fault in quadrotor aircraft may result in significant performance degradation. In order to achieve a reliable and accurate control of quadrotors, fault-tolerant capability should be enhanced in the design of the control system. With consideration of actuator faults and external disturbances, a state estimator was used to detect the actuator faults, and then a sliding mode controller was constructed to reject external disturbances and accommodate actuator faults in [9]. Based on an adaptive feedback linearization technique, the authors of [10] and [11] proposed the autonomous fault recovery schemes for a quadrotor in the presence of a certain fault in only one actuator and multiple faults in the actuators, respectively. With the use of a double control loop architecture, the control problem in case of failure of one rotor was addressed for the quadrotor aircraft, where the inner control loop handled the failure of one rotor, and the outer control loop enabled the vehicle to land safely at an arbitrary location [12].

In this paper, we consider the attitude tracking problem of quadrotors in the presence of external disturbances and multiple partial loss of rotor effectiveness faults. Similar to the existing approach in [1], the control scheme consists of two parts. In the first part, the desired attitude control torque for attitude tracking is designed. Since the rotations of the aircraft and the four rotors are combined together, an additional gyroscopic term should be contained in the attitude dynamics of the quadrotor. In addition, it is well known that there exist different kinds of uncertainties in the quadrotor aircraft, such as gust, ground effects and alteration of the engine torques, a time-varying but norm-bounded disturbance torque is also considered in the attitude dynamical model. Despite the external disturbances and the gyroscopic term, a robust controller is developed to achieve stable attitude trajectory tracking for a quadrotor by employing an indirect adaptive control technique. In the second part, a fault-tolerant motor torque controller is proposed to not only obtain the desired attitude tracking control torque designed in the first part, but also handle multiple partial loss of rotor effectiveness faults. A benefit of the proposed fault-tolerant controller is that it can reduce the fault effect without relying on the fault detection and diagnosis (FDD) mechanism. The stabilities of the attitude tracking controller in first part and

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fault-tolerant motor torque controller in second part can be guaranteed by the Lyapunov theory. In contrast to the attitude control control law in [1], the robustness to external disturbances and the fault-tolerant capability to rotor faults are enhanced in the proposed scheme.

The paper is organised as follows. In Section II, the nonlinear model of the quadrotor aircraft is introduced. The attitude tracking problem and the mathematical formulation of the control laws are presented in Sections III and Section IV, respectively. Simulation results are given in Section V to illustrate the feasibility of the proposed scheme. Conclusions are presented at the end of the paper.

## II. Mathematical Model

In this paper, the quadrotor aircraft is modeled as a rigid body with four rotors. As shown in Fig. 1, the front and the rear rotors rotate counterclockwise while the other two rotors rotate clockwise. Let $\mathcal{F}_i = \{\boldsymbol{x}_i, \boldsymbol{y}_i, \boldsymbol{z}_i\}$ denotes the inertial frame, and $\mathcal{F}_b = \{\boldsymbol{x}_b, \boldsymbol{y}_b, \boldsymbol{z}_b\}$ denotes the body-fixed frame of the quadrotor, where the origin of $\mathcal{F}_b$ is fixed to the center of mass of the quadrotor. The vectors $\boldsymbol{p} \in \mathbb{R}^3$ and $\boldsymbol{v} \in \mathbb{R}^3$ denote the position and linear velocity of the aircraft of the body-fixed frame $\mathcal{F}_b$ with respect to the inertial frame $\mathcal{F}_i$, respectively. In order to describe the attitude of the quadrotor without geometric singularity, the unit-quaternion representation is used. The unit-quaternion $\boldsymbol{Q} = [\boldsymbol{q}^T, q_0]^T$ contains a vector component $\boldsymbol{q} \in \mathbb{R}^3$ and a scalar component $q_0 \in \mathbb{R}$, subjecting to the constraint $\boldsymbol{q}^T\boldsymbol{q} + q_0^2 = 1$. The rotation matrix $\boldsymbol{R}(\boldsymbol{Q})$, related to the unit-quaternion, is given by $\boldsymbol{R}(\boldsymbol{Q}) = (q_0^2 - \boldsymbol{q}^T\boldsymbol{q})\boldsymbol{I}_3 + 2\boldsymbol{q}\boldsymbol{q}^T - 2q_0\boldsymbol{S}(\boldsymbol{q})$, where $\boldsymbol{I}_3$ is the 3-by-3 identity matrix, the matrix $\boldsymbol{S}(\boldsymbol{x})$ denotes a skew-symmetric matrix expressed in a certain frame such that $\boldsymbol{S}(\boldsymbol{x})\boldsymbol{\eta} = \boldsymbol{x} \times \boldsymbol{\eta}$ for any vectors $\boldsymbol{x} \in \mathbb{R}^3$ and $\boldsymbol{\eta} \in \mathbb{R}^3$ expressed in the same frame, where $\times$ denotes the cross product operation.

The dynamical model of a quadrotor aircraft can be described as follows [1]:

$$\begin{cases} (\Sigma_1): \begin{cases} \dot{\boldsymbol{p}} = \boldsymbol{v} \\ \dot{\boldsymbol{v}} = g\boldsymbol{z}_i - \frac{T}{m}\boldsymbol{R}^T(\boldsymbol{Q})\boldsymbol{z}_i \end{cases} \\ (\Sigma_2): \begin{cases} \dot{\boldsymbol{Q}} = \frac{1}{2}\begin{bmatrix} q_0\boldsymbol{I}_3 + \boldsymbol{S}(\boldsymbol{q}) \\ -\boldsymbol{q}^T \end{bmatrix}\boldsymbol{\omega} \\ \boldsymbol{J}_f\dot{\boldsymbol{\omega}} = -\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{J}_f\boldsymbol{\omega} - \boldsymbol{G_a} + \boldsymbol{\tau} + \boldsymbol{T_d} \end{cases} \\ (\Sigma_3): J_r\dot{\Omega}_i = u_i - M_i, \quad i \in \{1,2,3,4\}, \end{cases} \tag{1}$$

where $m$ is the mass of the aircraft, $g$ refers to the acceleration due to gravity, the scalar $T \in \mathbb{R}$ is the magnitude of the total lift force generated by the four rotors in the direction of $\boldsymbol{z}_i$, $\boldsymbol{\omega} \in \mathbb{R}^3$ is the angular velocity of the vehicle expressed in the body-fixed frame $\mathcal{F}_b$, $\boldsymbol{J}_f \in \mathbb{R}^3$ denotes the a symmetric positive definite constant inertia matrix of the vehicle with respect to the frame $\mathcal{F}_b$, $\boldsymbol{G_a}$ denotes the gyroscopic torque and is given by $\boldsymbol{G_a} = \sum_{i=1}^{4} J_r(\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{z}_i)(-1)^{i+1}\Omega_i$, the vector $\boldsymbol{\tau} \in \mathbb{R}^3$ and $\boldsymbol{T_d} \in \mathbb{R}^3$ represent the quadrotor rotation torques and the external disturbances expressed in $\mathcal{F}_b$, respectively. The motor torque and speed of the rotor $i$ are respectively represented by $u_i$ and $\Omega_i$, and the moment of inertia of each rotor is $J_r$. The reactive torque due to rotor drag of the $i$th rotor is denoted by $M_i$, which may be approximated as $M_i = k_a\Omega_i^2$ in free air with a positive drag factor $k_a$.

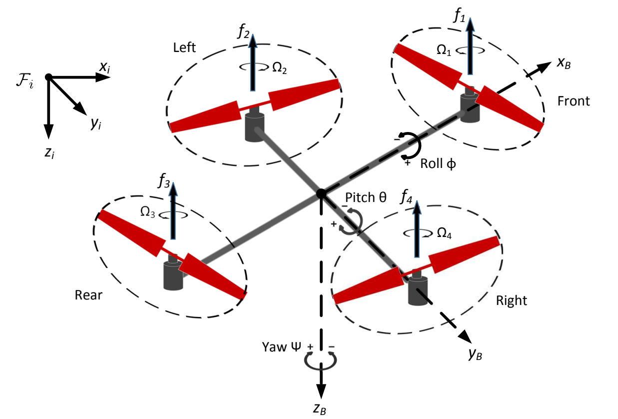



Fig. 1: Quadrotor aircraft configuration.

In the following, the relations between the rotation torques and the rotor velocities of the quadrotor aircraft are introduced. The thrust force generated by each rotor expressed in the body-fixed frame $\mathcal{F}_b$ in free air is given by

$$f_i = -b\Omega_i^2, \tag{2}$$

where $b > 0$ is the thrust factor. As a result, the total thrust generated by four rotors can be expressed as

$$T = \sum_{i=1}^{4} |f_i| = b\sum_{i=1}^{4} \Omega_i^2. \tag{3}$$

Based on the configuration of the quadrotor aircraft in Fig. 1, the rotation torques applied on the vehicle's body are given by $\boldsymbol{\tau} = (\tau_\phi, \tau_\theta, \tau_\psi)^T$ as follows:

$$\boldsymbol{\tau} = \begin{bmatrix} lb(\Omega_2^2 - \Omega_4^2) \\ lb(\Omega_1^2 - \Omega_3^2) \\ k_a(\Omega_1^2 + \Omega_3^2 - \Omega_2^2 - \Omega_4^2) \end{bmatrix}, \tag{4}$$

where $l$ is the distance from the center of mass to each rotor.

## III. Problem Formulation

The objective of this work is to design a feedback control law to accomplish the attitude tracking of the quadrotor aircraft subject to external disturbances and multiple partial loss of rotor effectiveness faults. For this purpose, the attitude dynamics $(\Sigma_2)$ and the rotor dynamics $(\Sigma_3)$ in (1) are considered. To formulate the fault-tolerant attitude tracking problem of the quadrotor, the attitude error dynamics with external disturbances and the rotor faults should be established first.

### A. Attitude Error Dynamics

Assuming that the desired quadrotor attitude $\boldsymbol{Q}_d$ and angular velocity $\boldsymbol{\omega}_d$ have been determined by designing the total thrust $T$ of the translational dynamics $(\Sigma_1)$ in (1), the attitude tracking error $\tilde{\boldsymbol{Q}} = [\tilde{\boldsymbol{q}}^T, \tilde{q}_0]^T$ is defined as the relative orientation between the attitude $\boldsymbol{Q}$ and the desired attitude $\boldsymbol{Q}_d$, which is computed as

$$\tilde{\boldsymbol{Q}} = \boldsymbol{Q}_d^{-1} \otimes \boldsymbol{Q}, \tag{5}$$

where $\boldsymbol{Q}_d^{-1}$ is the inverse or conjugate of the desired quaternion and is determined by $\boldsymbol{Q}_d^{-1} = [-\boldsymbol{q}_d^T, q_{d0}]^T$, and $\otimes$ denotes the quaternion multiplication operator of two unit quaternion $\boldsymbol{Q}_i = [\boldsymbol{q}_i^T, q_{i0}]^T$ and $\boldsymbol{Q}_j = [\boldsymbol{q}_j^T, q_{j0}]^T$, which is defined as follows:

$$\boldsymbol{Q}_i \otimes \boldsymbol{Q}_j = \begin{bmatrix} q_{i0}\boldsymbol{q}_j + q_{j0}\boldsymbol{q}_i + \boldsymbol{S}(\boldsymbol{q}_i)\boldsymbol{q}_j \\ q_{i0}q_{j0} - \boldsymbol{q}_i^T\boldsymbol{q}_j \end{bmatrix}. \tag{6}$$

The angular velocity error $\tilde{\boldsymbol{\omega}} \in \mathbb{R}^3$ is represented by $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega} - \boldsymbol{R}(\tilde{\boldsymbol{Q}})\boldsymbol{\omega}_d$, where $\boldsymbol{R}(\tilde{\boldsymbol{Q}})$ is the rotation matrix related to $\tilde{\boldsymbol{Q}}$, and satisfies $||\boldsymbol{R}(\tilde{\boldsymbol{Q}})|| = 1$ and $\dot{\boldsymbol{R}}(\tilde{\boldsymbol{Q}}) = -\boldsymbol{S}(\tilde{\omega})\boldsymbol{R}(\tilde{\boldsymbol{Q}})$. Thus, based on the attitude dynamics $(\Sigma_2)$, the attitude tracking error dynamics can be derived as

$$\begin{cases} \dot{\tilde{\boldsymbol{Q}}} = \frac{1}{2}\begin{bmatrix} \tilde{q}_0\boldsymbol{I}_3 + \boldsymbol{S}(\tilde{\boldsymbol{q}}) \\ -\tilde{\boldsymbol{q}}^T \end{bmatrix}\tilde{\boldsymbol{\omega}} \\ \boldsymbol{J}_f\dot{\tilde{\boldsymbol{\omega}}} = \boldsymbol{J}_f\left(\boldsymbol{S}(\tilde{\omega})\boldsymbol{R}(\tilde{\boldsymbol{Q}})\boldsymbol{\omega}_d - \boldsymbol{R}(\tilde{\boldsymbol{Q}})\dot{\boldsymbol{\omega}}_d\right) \\ \qquad -\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{J}_f\boldsymbol{\omega} - \boldsymbol{G_a} + \boldsymbol{\tau} + \boldsymbol{T_d} \end{cases} \tag{7}$$

*Assumption 1:* The external disturbance $\boldsymbol{T_d}$ is bounded such that $||\boldsymbol{T_d}|| \leq d_{max}$, where $d_{max}$ is a positive constant and $||\cdot||$ denotes the Euclidean norm.

*Assumption 2:* The desired angular velocity $\boldsymbol{\omega}_d$ and its derivative $\dot{\boldsymbol{\omega}}_d$ are bounded such that $||\boldsymbol{\omega}_d|| \leq c_1$ and $||\dot{\boldsymbol{\omega}}_d|| \leq c_2$ for all $t \geq 0$, where $c_1 \geq 0$ and $c_2 \geq 0$ are two finite unknown constants.

*Assumption 3:* The symmetric positive-definitive inertia matrix $\boldsymbol{J}_f$ is upper bounded, i.e., there exists a positive constant $c_{\boldsymbol{J}_f} > 0$ such that $||\boldsymbol{J}_f|| \leq c_{\boldsymbol{J}_f}$.

### B. Rotor Fault Model

In case of partial loss of effective faults in actuator, the motor torque $u_i$ in rotor dynamics $(\Sigma_3)$ of (1) becomes different from the designed value. Let $u_i^F$ denotes the faulty motor torque of the $i$th rotor, and then the rotor fault model can be described as follows:

$$u_i^F = e_i u_i, \quad 0 < \epsilon_i \leq e_i \leq 1 \tag{8}$$

where $e_i$ stands for the effectiveness of the $i$th rotor, $\epsilon_i$ is a known lower bound of $e_i$ satisfying $0 < \epsilon_i \leq 1$, and $i \in \{1,2,3,4\}$. Note that the case $e_i = 1$ indicates that the $i$th actuator works normally, and $\epsilon \leq e_i < 1$ implies that the $i$th actuator partially loses its effectiveness, but still has not totally failed.

According to (8), one can define $\boldsymbol{u}^F = [u_1^F, u_2^F, u_3^F, u_4^F]^T = \boldsymbol{E}\boldsymbol{u}$ with $\boldsymbol{E} = diag[e_1, e_2, e_3, e_4] \in \mathbb{R}^{4\times4}$ and $\boldsymbol{u} = [u_1, u_2, u_3, u_4]^T \in \mathbb{R}^4$, then the rotor dynamics with rotor faults (8) is given by

$$(\Sigma_3^f) : \boldsymbol{J}_r\dot{\boldsymbol{\Omega}} = \boldsymbol{E}\boldsymbol{u} - \boldsymbol{M}, \tag{9}$$

where $\boldsymbol{J}_r = J_r\boldsymbol{I}_4 \in \mathbb{R}^{4\times4}$, $\boldsymbol{\Omega} = [\Omega_1, \Omega_2, \Omega_3, \Omega_4]^T \in \mathbb{R}^4$, and $\boldsymbol{M} = [M_1, M_2, M_3, M_4]^T \in \mathbb{R}^4$.

## IV. Attitude Tracking Controller Design

In order to solve the attitude tracking problem of the quadrotor aircraft under external disturbances and partial loss of rotor effectiveness faults, similar to the approach in [1], two steps are involved in the control scheme. In the first step, disturbances rejection and stable attitude trajectory tracking are achieved through a robust controller on the basis of an indirect adaptive control method. Then, the obtained rotation torques in the first part are regarded as the desired attitude tracking torques, and a fault-tolerant motor torque controller is developed in the second step to achieve this desired torques with implicit consideration of rotor faults.

### A. Step 1: Attitude Tracking Torques Design

Based on the attitude tacking error dynamics in (7), an indirect robust adaptive controller is designed to track the desired attitude of a rigid quadrotor in the presence of external disturbances in this part. To design the attitude tracking controller, the following auxiliary variable is introduced [13]

$$\boldsymbol{z} = \tilde{\boldsymbol{\omega}} + k\tilde{\boldsymbol{q}}, \tag{10}$$

where $k$ is a positive constant. Then, from the attitude tacking error dynamics in (7), it can be derived that

$$\boldsymbol{J}_f\dot{\boldsymbol{z}} = \boldsymbol{f}\left(\boldsymbol{\omega}, \boldsymbol{\omega}_d, \dot{\boldsymbol{\omega}}_d, \boldsymbol{Q}, \boldsymbol{Q}_d, \boldsymbol{T_d}\right) - \boldsymbol{G}_a + \boldsymbol{\tau}, \tag{11}$$

where $\boldsymbol{f}\left(\boldsymbol{\omega}, \boldsymbol{\omega}_d, \dot{\boldsymbol{\omega}}_d, \boldsymbol{Q}, \boldsymbol{Q}_d, \boldsymbol{T_d}\right)$ denotes the a lumped term containing the system nonlinearities and external disturbances, and is defined as

$$\begin{aligned} \boldsymbol{f}\left(\boldsymbol{\omega}, \boldsymbol{\omega}_d, \dot{\boldsymbol{\omega}}_d, \boldsymbol{Q}, \boldsymbol{Q}_d, \boldsymbol{T_d}\right) &= \boldsymbol{J}_f\left(\boldsymbol{S}(\tilde{\omega})\boldsymbol{R}(\tilde{\boldsymbol{Q}})\boldsymbol{\omega}_d - \boldsymbol{R}(\tilde{\boldsymbol{Q}})\dot{\boldsymbol{\omega}}_d\right) \\ &- \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{J}_f\boldsymbol{\omega} + \frac{k}{2}\boldsymbol{J}_f\left(\tilde{q}_0\boldsymbol{I}_3 + \boldsymbol{S}(\tilde{\boldsymbol{q}})\right)\tilde{\boldsymbol{\omega}} + \boldsymbol{T_d}. \end{aligned} \tag{12}$$

By using the preceding assumptions and the facts that $||\tilde{q}_0\boldsymbol{I}_3 + \boldsymbol{S}(\tilde{\boldsymbol{q}})|| = 1$ [14], $||\boldsymbol{R}(\tilde{\boldsymbol{Q}})|| = 1$, and $||\tilde{\boldsymbol{Q}}|| = 1$, the following inequalities are established:

$$\begin{aligned} &||\boldsymbol{J}_f\left(\tilde{q}_0\boldsymbol{I}_3 + \boldsymbol{S}(\tilde{\boldsymbol{q}})\right)\tilde{\boldsymbol{\omega}}|| \leq c_{\boldsymbol{J}_f}(||\boldsymbol{\omega}|| + c_1) \\ &||\boldsymbol{J}_f(\boldsymbol{S}(\tilde{\omega})\boldsymbol{R}(\tilde{\boldsymbol{Q}})\boldsymbol{\omega}_d - \boldsymbol{R}(\tilde{\boldsymbol{Q}})\dot{\boldsymbol{\omega}}_d)|| \leq c_{\boldsymbol{J}_f}(c_1||\boldsymbol{\omega}|| + c_1^2 + c_2) \\ &||-\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{J}_f\boldsymbol{\omega} + \boldsymbol{T_d}|| \leq c_{\boldsymbol{J}_f}||\boldsymbol{\omega}||^2 + d_{max}. \end{aligned} \tag{13}$$

Thus, it yields

$$\begin{aligned} &||\boldsymbol{f}\left(\boldsymbol{\omega}, \boldsymbol{\omega}_d, \dot{\boldsymbol{\omega}}_d, \boldsymbol{Q}, \boldsymbol{Q}_d, \boldsymbol{T_d}\right)|| \leq c_{\boldsymbol{J}_f}||\boldsymbol{\omega}||^2 \\ &\quad + (\frac{k}{2}c_{\boldsymbol{J}_f} + c_{\boldsymbol{J}_f}c_1)||\boldsymbol{\omega}|| + \frac{k}{2}c_{\boldsymbol{J}_f}c_1 + c_{\boldsymbol{J}_f}(c_1^2 + c_2) + d_{max}. \end{aligned} \tag{14}$$

Furthermore, it can conclude that

$$||\boldsymbol{f}\left(\boldsymbol{\omega}, \boldsymbol{\omega}_d, \dot{\boldsymbol{\omega}}_d, \boldsymbol{Q}, \boldsymbol{Q}_d, \boldsymbol{T_d}\right)|| \leq cH, \tag{15}$$

where $c = \max\{c_{\boldsymbol{J}_f}, \frac{k}{2}c_{\boldsymbol{J}_f} + c_{\boldsymbol{J}_f}c_1, \frac{k}{2}c_{\boldsymbol{J}_f}c_1 + c_{\boldsymbol{J}_f}(c_1^2 + c_2) + d_{max}\}$, and $H$ is given by $H = ||\boldsymbol{\omega}||^2 + ||\boldsymbol{\omega}|| + 1$.

Now, the attitude tracking controller for the attitude dynamics are proposed as follows:

*Theorem 1:* Consider the attitude control systems $(\Sigma_2)$ in (1), the attitude tracking controller for the quadrotor is designed as

$$\boldsymbol{\tau} = \boldsymbol{G}_a - \left(k_0 + k_z \frac{\hat{c}H^2}{H||\boldsymbol{z}|| + \sigma}\right)\boldsymbol{z}, \tag{16}$$

where $k_0$, $\sigma$, and $k_z \geq 1$ are positive constants, and $\hat{c}$ is updated by

$$\dot{\hat{c}}(t) = -l_1(l_2\hat{c}(t) - H||\boldsymbol{z}||), \tag{17}$$

with $\hat{c}(0) \geq 0$ to guarantee $\hat{c}(t) \geq 0$. Then, the angular velocity and attitude tracking errors of the quadrotor converge to a small set containing the origin.

*Proof:* Consider the following Lyapunov candidate

$$V_a = \frac{1}{2}\boldsymbol{z}^T\boldsymbol{J}_f\boldsymbol{z} + 2k_0k\left[\tilde{\boldsymbol{q}}^T\tilde{\boldsymbol{q}} + (1-\tilde{q}_0)^2\right] + \frac{1}{2l_1}\tilde{c}^2(t) \tag{18}$$

where $\tilde{c}(t) = \hat{c}(t) - c$. The time derivative of $V_a$ along the trajectory of (11) is given as

$$\begin{aligned}\dot{V}_a =& \boldsymbol{z}^T\left(\boldsymbol{f}\left(\boldsymbol{\omega}, \boldsymbol{\omega}_d, \dot{\boldsymbol{\omega}}_d, \boldsymbol{Q}, \boldsymbol{Q}_d, \boldsymbol{T_d}\right) - \boldsymbol{G}_a + \boldsymbol{\tau}\right)\\ &+ 2k_0k\tilde{\boldsymbol{q}}^T\tilde{\boldsymbol{\omega}} + \frac{1}{l_1}\tilde{c}(t)\dot{\hat{c}}(t).\end{aligned} \tag{19}$$

In view of the control law in (16) and adaptive law in (17) , and using the inequality in (15), it can be shown that

$$\begin{aligned}\dot{V}_a =& \boldsymbol{z}^T\boldsymbol{f}\left(\boldsymbol{\omega}, \boldsymbol{\omega}_d, \dot{\boldsymbol{\omega}}_d, \boldsymbol{Q}, \boldsymbol{Q}_d, \boldsymbol{T_d}\right) - k_0||\boldsymbol{z}||^2 + 2k_0k\tilde{\boldsymbol{q}}^T\tilde{\boldsymbol{\omega}}\\ &- k_z\frac{\hat{c}(t)H^2}{H||\boldsymbol{z}|| + \sigma}||\boldsymbol{z}||^2 - \tilde{c}(t)(l_2\hat{c}(t) - H||\boldsymbol{z}||)\\ \leq& - k_0||\boldsymbol{z}||^2 + 2k_0k\tilde{\boldsymbol{q}}^T\tilde{\boldsymbol{\omega}} - l_2(\hat{c}(t)^2 - c\hat{c}(t))\\ &+ \frac{(c + \tilde{c}(t) - k_z\hat{c}(t))H^2||\boldsymbol{z}||^2 + \sigma(c - \tilde{c}(t))H||\boldsymbol{z}||}{H||\boldsymbol{z}|| + \sigma}\\ =& - k_0||\boldsymbol{z}||^2 + 2k_0k\tilde{\boldsymbol{q}}^T\tilde{\boldsymbol{\omega}} - l_2(\hat{c}(t)^2 - c\hat{c}(t))\\ &+ \frac{(1 - k_z)\hat{c}(t)H^2||\boldsymbol{z}||^2 + \sigma\hat{c}(t)H||\boldsymbol{z}||}{H||\boldsymbol{z}|| + \sigma}.\end{aligned} \tag{20}$$

From $k_z \geq 1$, $\hat{c}(t) \geq 0$, and the fact that $\frac{H||\boldsymbol{z}||}{H||\boldsymbol{z}||+\sigma} < 1$, $(\forall H > 0, \sigma > 0)$, we have

$$\begin{aligned}\dot{V}_a \leq& - k_0||\boldsymbol{z}||^2 + 2k_0k\tilde{\boldsymbol{q}}^T\tilde{\boldsymbol{\omega}} - l_2\hat{c}^2(t) + (l_2c + \sigma)\hat{c}(t)\\ =& - k_0||\tilde{\boldsymbol{\omega}}||^2 - k_0k^2||\tilde{\boldsymbol{q}}||^2\\ &- l_2(\hat{c}(t) - \frac{l_2c + \sigma}{2l_2})^2 + \frac{(l_2c + \sigma)^2}{4l_2}\\ \leq& - k_0||\tilde{\boldsymbol{\omega}}||^2 - k_0k^2||\tilde{\boldsymbol{q}}||^2 + \frac{(l_2c + \sigma)^2}{4l_2}.\end{aligned} \tag{21}$$

Thus, $\dot{V}_a$ is strictly negative when $\tilde{\boldsymbol{\omega}}$ or $\tilde{\boldsymbol{q}}$ is, respectively, outside of the compact set $\Delta_1 = \left\{\tilde{\boldsymbol{\omega}}(t)\,\middle|\,||\tilde{\boldsymbol{\omega}}(t)|| \leq (l_2c + \sigma)/\sqrt{4k_0l_2}\right\}$ or $\Delta_2 = \left\{\tilde{\boldsymbol{q}}(t)\,\middle|\,||\tilde{\boldsymbol{q}}(t)|| \leq (l_2c + \sigma)/\sqrt{4k_0k^2l_2}\right\}$, which implies that $V_a$ decreases monotonically. The decrease of $V_a$ eventually drives $\tilde{\boldsymbol{\omega}}$ and $\tilde{\boldsymbol{q}}$ into the compact set $\Delta_1$ and $\Delta_2$ according to (18) and (21). Therefore, the angular velocity and attitude tracking errors are bounded to a compact set including the origin ultimately. This ends the proof. ■

### B. Step 2: Fault-Tolerant Motor Torques Design

Since the four rotors of quadrotor aircraft are actually driven by the motor torque $u_i$, the designed attitude tracking torques $\boldsymbol{\tau}$ and the total thrust $T$ should be maintained by specifying the desired speed of each rotor $\boldsymbol{\Omega}_d = [\Omega_{d,1}, \Omega_{d,2}, \Omega_{d,3}, \Omega_{d,4}]^T$. Then, the commanded motor torques are designed such that the speed of each rotor can converge to their desired speed. From (3) and (4), it is possible to show that

$$\begin{bmatrix} T \\ \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} = \begin{bmatrix} b & b & b & b \\ 0 & lb & 0 & -lb \\ lb & 0 & -lb & 0 \\ k_a & -k_a & k_a & -k_a \end{bmatrix} \begin{bmatrix} \Omega_{d,1}^2 \\ \Omega_{d,2}^2 \\ \Omega_{d,3}^2 \\ \Omega_{d,4}^2 \end{bmatrix}, \tag{22}$$

the above $4 \times 4$ coefficient matrix is always invertible as long as $lbk_a \neq 0$. Thus, for given lift force magnitude and given attitude tracking control torques, the desired speed of the four rotors can be obtained from (22).

When partial loss of actuator effectiveness fault is considered in the rotor dynamics, the commanded motor torques should be designed with fault-tolerance capacity. In order to get the online information of the rotor effectiveness, an observer to the faulty rotor dynamics described in (9) is introduced as follows:

$$\boldsymbol{J}_r\dot{\hat{\boldsymbol{\Omega}}} = \hat{\boldsymbol{E}}(t)\boldsymbol{\nu} - \boldsymbol{M}, \tag{23}$$

where $\hat{\boldsymbol{\Omega}} = [\hat{\Omega}_1, \hat{\Omega}_2, \hat{\Omega}_3, \hat{\Omega}_4]^T \in \mathbb{R}^4$ is the estimated speed of rotors, $\hat{\boldsymbol{E}} = diag[\hat{e}_1(t), \hat{e}_2(t), \hat{e}_3(t), \hat{e}_4(t)] \in \mathbb{R}^{4\times4}$ is the estimated effectiveness matrix of rotors, and the input $\boldsymbol{\nu} = [\nu_1, \nu_2, \nu_3, \nu_4]^T \in \mathbb{R}^4$ of the observed faulty dynamics will be given later.

Define the observation error vector as $\tilde{\boldsymbol{\Omega}} = [\tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3, \tilde{\Omega}_4]^T \in \mathbb{R}^4$ with $\tilde{\Omega}_i = \hat{\Omega}_i - \Omega_i$, $i \in \{1, 2, 3, 4\}$, and choose the motor torque $\boldsymbol{u} = \boldsymbol{\nu} + k_u\tilde{\boldsymbol{\Omega}}$, then the observation error dynamics is given as

$$\boldsymbol{J}_r\dot{\tilde{\boldsymbol{\Omega}}} = \tilde{\boldsymbol{E}}(t)\boldsymbol{\nu} - k_u\boldsymbol{E}\tilde{\boldsymbol{\Omega}}, \tag{24}$$

where $\tilde{\boldsymbol{E}}(t) = diag[\tilde{e}_1(t), \tilde{e}_2(t), \tilde{e}_3(t), \tilde{e}_4(t)] \in \mathbb{R}^{4\times4}$ is the estimated effectiveness error with $\tilde{e}_i(t) = \hat{e}_i(t) - e_i$, $k_u$ is a positive constant. To design an input $\boldsymbol{\nu}$ such that the estimated speed of rotors $\hat{\boldsymbol{\Omega}}$ can track the desired speed of rotors $\boldsymbol{\Omega}_d$, the observer tracking error vector is defined as $\hat{\boldsymbol{\Omega}}_e = [\hat{\Omega}_{e,1}, \hat{\Omega}_{e,2}, \hat{\Omega}_{e,3}, \hat{\Omega}_{e,4}]^T \in \mathbb{R}^4$ with $\hat{\Omega}_{e,i} = \hat{\Omega}_i - \Omega_{d,i}$, $i \in \{1, 2, 3, 4\}$. Using (23), the observer tracking error dynamics is written as

$$\boldsymbol{J}_r\dot{\hat{\boldsymbol{\Omega}}}_e = \hat{\boldsymbol{E}}(t)\boldsymbol{\nu} - \boldsymbol{M} - \boldsymbol{J}_r\dot{\boldsymbol{\Omega}}_d. \tag{25}$$

Now, one can state the following result.

*Theorem 2:* Consider the faulty rotor dynamics $(\Sigma_3^f)$ given in (9) with partial loss of effectiveness fault, the motor torques are designed as

$$\boldsymbol{u} = \boldsymbol{\nu} + k_u\tilde{\boldsymbol{\Omega}}, \tag{26}$$

where $\boldsymbol{\nu} = \hat{\boldsymbol{E}}^{-1}(t)(-k_v\hat{\boldsymbol{\Omega}}_e + \boldsymbol{M} + \boldsymbol{J}_r\dot{\boldsymbol{\Omega}}_d)$ is the input of the observer (23), $k_v$ is a positive constant, and $\hat{e}_i(t)$ is updated according to the adaptive low

$$\begin{aligned}\dot{\hat{e}}_i(t) &= \mathrm{Proj}_{[\epsilon_i,1]}\{-\alpha_i\tilde{\Omega}_i\nu_i\} \\ &= \begin{cases} 0, & \text{if } \hat{e}_i(t)=\epsilon_i, -\alpha_i\tilde{\Omega}_i\nu_i \le 0 \text{ or} \\ & \hat{e}_i(t)=1, -\alpha_i\tilde{\Omega}_i\nu_i \ge 0 \\ -\alpha_i\tilde{\Omega}_i\nu_i, & \text{otherwise} \end{cases}\end{aligned} \tag{27}$$

where $\alpha_i$ is the positive adaptive gain, and the projection operator $\mathrm{Proj}\{\cdot\}$ is used to keep the parameter estimate within the parameter bound. Then, the speed of each rotor $\Omega_i$ can converge to their desired speed $\Omega_{d,i}$ asymptotically, and the designed attitude tracking torques can be maintained despite partial loss of rotor effectiveness fault.

*Proof:* Consider the candidate Lyapunov function

$$V_r = \frac{1}{2}\hat{\boldsymbol{\Omega}}_e^T \boldsymbol{J}_r \hat{\boldsymbol{\Omega}}_e + \frac{1}{2}\tilde{\boldsymbol{\Omega}}^T \boldsymbol{J}_r \tilde{\boldsymbol{\Omega}} + \frac{1}{2}\sum_{i=1}^{4}\frac{\tilde{e}_i^2(t)}{\alpha_i}. \tag{28}$$

The time derivative of the Lyapunov function in view of (24) and (25) satisfies

$$\begin{aligned}\dot{V}_r \le & -k_v||\hat{\boldsymbol{\Omega}}_e||^2 - k_u\tilde{\boldsymbol{\Omega}}^T\boldsymbol{E}\tilde{\boldsymbol{\Omega}} \\ & + \tilde{\boldsymbol{\Omega}}^T\tilde{\boldsymbol{E}}(t)\boldsymbol{\nu} + \sum_{i=1}^{4}\frac{\tilde{e}_i(t)\dot{\tilde{e}}_i(t)}{\alpha_i}.\end{aligned} \tag{29}$$

Since $e_i$ is an unknown constant, it follows that $\dot{\tilde{e}}_i(t) = \dot{\hat{e}}_i(t)$. Then, based on the property of projection operator [15], it can be found that

$$\frac{\tilde{e}_i(t)\dot{\tilde{e}}_i(t)}{\alpha_i} \le -\tilde{e}_i(t)\tilde{\Omega}_i\nu_i. \tag{30}$$

Let $\gamma = \lambda_{\min}\{\boldsymbol{E}\}$, where $\lambda_{\min}\{\cdot\}$ denotes the minimum eigenvalue of a matrix. Since $e_i$ satisfies $0 < \epsilon_i \le e_i \le 1$, it is clear that $\gamma > 0$. Further simplification of (29), using the adaptive law in (27), leads to

$$\begin{aligned}\dot{V}_r \le & -k_v||\hat{\boldsymbol{\Omega}}_e||^2 - k_u\gamma||\tilde{\boldsymbol{\Omega}}||^2 + \sum_{i=1}^{4}\left(\tilde{\Omega}_i\tilde{e}_i\nu_i + \frac{\tilde{e}_i(t)\dot{\tilde{e}}_i(t)}{\alpha_i}\right) \\ \le & -k_v||\hat{\boldsymbol{\Omega}}_e||^2 - k_u\gamma||\tilde{\boldsymbol{\Omega}}||^2.\end{aligned} \tag{31}$$

Therefore, $\dot{V}_r$ is negative semi-definite. Hence, it can be concluded that $\hat{\boldsymbol{\Omega}}_e \in \mathcal{L}_\infty$ and $\tilde{\boldsymbol{\Omega}} \in \mathcal{L}_\infty$. Upon integrating $\dot{V}_r$ from 0 to $\infty$, one obtains

$$\begin{aligned}V_r(0) - V_r(\infty) \ge & k_v\int_0^\infty ||\hat{\boldsymbol{\Omega}}_e(\zeta)||^2 d\zeta \\ & + k_u\gamma\int_0^\infty ||\tilde{\boldsymbol{\Omega}}(\zeta)||^2 d\zeta.\end{aligned} \tag{32}$$

Since the term on the left-hand side of the above inequality is bounded, it follows that $\hat{\boldsymbol{\Omega}}_e \in \mathcal{L}_\infty \cap \mathcal{L}_2$ and $\tilde{\boldsymbol{\Omega}} \in \mathcal{L}_\infty \cap \mathcal{L}_2$. In addition, from (24) and (25), one can easily verify that $\dot{\hat{\boldsymbol{\Omega}}}_e \in \mathcal{L}_\infty$ and $\dot{\tilde{\boldsymbol{\Omega}}} \in \mathcal{L}_\infty$. Consequently, by invoking Barbalat's lemma, it yields that

$$\lim_{t\to\infty}\hat{\boldsymbol{\Omega}}_e(t) = \lim_{t\to\infty}\tilde{\boldsymbol{\Omega}}(t) = 0, \tag{33}$$

which implies that $\lim_{t\to\infty}(\boldsymbol{\Omega}(t) - \boldsymbol{\Omega}_d(t)) = 0$. Therefore, the speed of rotor can converge to their desired value asymptotically. As a result, the designed attitude tracking torques in step 1 are achieved in spite of the occurrence of partial loss of rotor effectiveness faults. ■

TABLE I: Quadrotor aircraft model parameters

| | Value | Unit | Description |
|---|---|---|---|
| $m$ | 0.468 | $kg$ | Mass |
| $g$ | 9.81 | $m/s^2$ | Gravity acceleration |
| $J_{f_\phi}$ | $4.9 \times 10^{-3}$ | $kg \cdot m^2$ | Inertia coefficient along roll axis |
| $J_{f_\theta}$ | $4.9 \times 10^{-3}$ | $kg \cdot m^2$ | Inertia coefficient along pitch axis |
| $J_{f_\psi}$ | $8.8 \times 10^{-3}$ | $kg \cdot m^2$ | Inertia coefficient along yaw axis |
| $J_r$ | $3.4 \times 10^{-5}$ | $kg \cdot m^2$ | Inertia of rotor |
| $l$ | 0.225 | $m$ | Distance between rotor and c.g. |

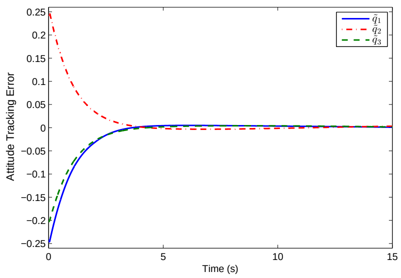
TABLE II: Numerical Simulation Parameters

| Parameter name | Value |
|---|---|
| Initial attitude | $\boldsymbol{Q}(0) = [-0.1, 0.15, -0.2, 0.96]^T$ |
| Initial angular velocity | $\boldsymbol{\omega}(0) = [0, 0, 0]^T$ |
| Initial linear velocity | $\boldsymbol{v}(0) = [0.1, 0.1, 0]^T$ |
| Initial speed of rotors | $\boldsymbol{\Omega}(0) = [100, 100, 100, 100]^T$ |
| Controller parameters of attitude tracking torques design | $k = 2$, $k_0 = 0.1$, $k_z = 0.1$, $l_1 = 0.2$, $l_2 = 0.5$, $\sigma = 0.05$, $\hat{c}(0) = 0$ |
| Controller parameters of rotor torques design | $k_u = 0.008$, $k_v = 0.002$, $\alpha_i = 2$, $\hat{e}_i(0) = 1$, $i \in \{1, 2, 3, 4\}$ |

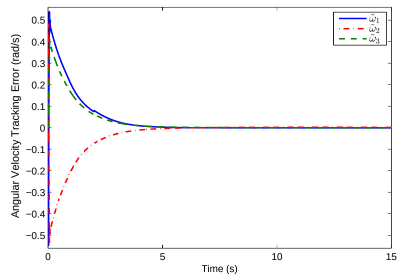
## V. Simulation Results

In this section, simulation results are presented to illustrate the effectiveness of the proposed attitude tracking control scheme under partial loss of actuator effectiveness faults. The quadrotor aircraft model parameters are given in Table I. Consider the translational dynamics $(\Sigma_1)$ in (1), the total lift thrust $\boldsymbol{T}$ is designed to control the position of the quadrotor through a PD controller. Then, we can obtain the necessary thrust amplitude $T$ and the desired attitude $\boldsymbol{Q}_d$ of the quadrotor. The initial position of the quadrotor aircraft is $\boldsymbol{p}(0) = [0, 0, 10]^T$, and the target position is $\boldsymbol{p}_d(0) = [1, 1, 10]^T$ in the inertial frame. The external disturbances $\boldsymbol{T_d}$ in (1) are assumed to be $\boldsymbol{T_d} = 10^{-3} \times [8\cos(0.1t), -5\sin(0.3t), 6\sin(0.2t)]^T$ Nm. The actuator fault scenario considered here assumes that the first rotor undergoes a 10% loss of its effectiveness at $t = 1$ s, while at $t = 2$ s, the forth rotor experiences a 20% loss of its effectiveness. The other simulation parameters are given in Table II.

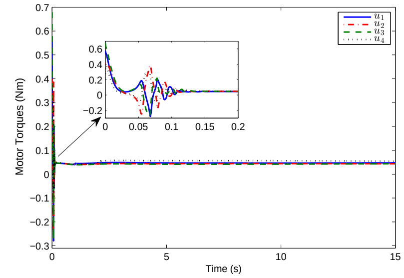
The simulation results are shown in Fig. 2. Figs. 2a and 2b show the attitude and angular velocity tracking errors, respectively. The attitude tracking torques are presented in Fig. 2f. It is clear from these figures that the angular velocity and attitude tracking errors converge to a small neighbourhood of the origin, and good control performance is obtained even in the presence of the external disturbances. Fig. 2d illustrates the observer tracking error and Fig. 2e illustrates the observation error. The motor torque histories are given in Fig. 2c. From Figs. 2d and 2e, both two error vectors can quickly converge to zero, which guarantees the tracking of



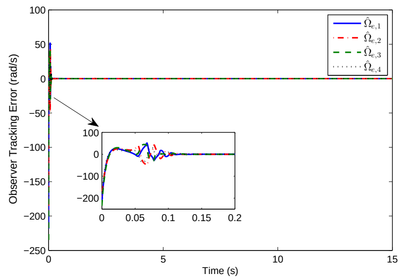

(a) Attitude tracking error.



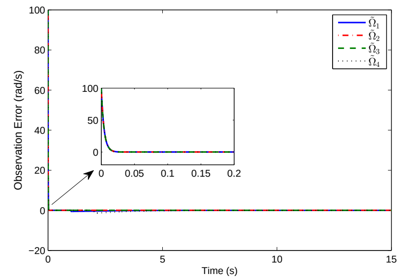

(b) Angular velocity tracking error (rad/s).



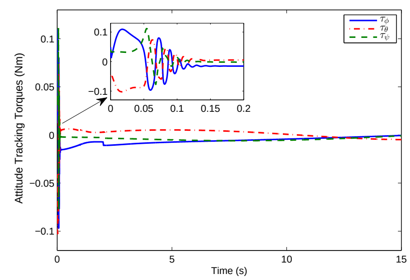

(c) Motor torques.




(d) Observer tracking error.




(e) Observation error.




(f) Attitude tracking torques.

Fig. 2: Simulation results of a quadrotor under partial loss of rotor effectiveness faults.

the required rotor speeds. As a result, the attitude tracking torque commands can be realized by the rotor torques despite partial loss of rotor effectiveness faults.

## VI. Conclusion

In this paper, attitude tracking problem for a quadrotor against partial loss of rotor effectiveness faults using indirect robust adaptive control technique is addressed. In the attitude tracking controller design, unit quaternion is used to represent the attitude of the quadrotor aircraft, and external disturbances and gyroscopic term have been taken into account. The attitude tracking torques are developed to achieve stable attitude tracking and the tracking errors converge to a small set containing the origin. In the fault-tolerant motor torque controller design, based on the on-line estimation of the rotor effectiveness factor, an adaptive fault-tolerant controller is proposed to compensate the fault effects such that the designed attitude tracking toques are maintained by designing the motor torques despite the actuator faults. The simulation results show the effectiveness of the proposed fault-tolerant attitude tracking scheme.

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