Tractable Compositions of Discrete-Time Control Barrier Functions with Application to Driving Safety Control

Mohammad Khajenejad* Matthew Cavorsi* Ruochen Niu Qiang Shen Sze Zheng Yong

***Abstract*— This paper introduces control barrier functions for discrete-time systems, which can be shown to be necessary and sufficient for controlled invariance of a given set. In particular, we propose nonlinear discrete-time control barrier functions for control affine systems with an additional structure that lead to controlled invariance conditions that are affine in the control input, resulting in a tractable formulation that enables us to handle the safety optimal control problem for a broader range of applications with more complicated safety conditions than existing approaches. Moreover, we develop alternative mixed-integer formulations for basic and secondary Boolean compositions of multiple control barrier functions and further provide mixed-integer constraints for piecewise control barrier functions. Finally, we apply these proposed tools to driving safety problems of lane keeping and obstacle avoidance, which are shown to be effective in simulation.**

I. Introduction

Motivated by safety-critical applications such as adaptive cruise control systems [1], multi-agent systems [2] and footstep placement of bipedal robots [3], several control approaches have been developed to guarantee safety, in addition to addressing the stabilization problem. In particular, approaches based on set invariance using control barrier functions have lately garnered a lot of research attention.

Literature review. A variety of Lyapunov-like approaches have been developed to construct *barrier certificates* and (controlled) invariant sets for ensuring system safety, both for autonomous systems, e.g., in [4]–[6] and for control systems, e.g., [1]–[3], [7]–[10]. Moreover, these Control Barrier Functions (CBFs) can be combined with control Lyapunov functions, yielding Control Lyapunov Barrier Functions (CLBFs), which have been shown in recent years to be a promising approach for jointly guaranteeing safety and stability.

Although CBFs and CLBFs have been extensively studied in the control and verification literature for a broad range of continuous-time systems for applications such as model predictive control, obstacle/collision avoidance, eventuality properties or safety establishment and multiobjective control [1], [7], [11]–[17], there are only relatively few studies that address the design of CBF-based approaches for discrete-time dynamical systems (including sampled data and inherently discrete-time/digital systems). The work in [18] extends the continuous-time CBF-based developed tools for safety-critical applications to discrete-time systems, and established that the extension is not straightforward because the resulting optimization problem is not necessarily convex and hence *tractability* remains an unsolved issue, except for some special cases such as linear/linearized settings.

On the other hand, the authors in [19] developed a barrier function based model predictive control for a class of nonlinear discrete-time dynamics, which hinges extensively on the stabilizability of the linearized system, while [20] applied discrete-time barrier functions to ensure safety of a given set for multi-agent partially observable Markov decision processes and further proposed conditions for checking Boolean compositions of barrier functions to represent more complicated safety sets. However, the assumption of finite and countable actions (i.e., control inputs) as well as the Markov assumption are essential for obtaining a tractable solution in [20]; thus, this approach does not directly apply to the general discrete-time systems that we consider.

Contribution. In this paper, we present a necessary and sufficient formulation of control barrier functions for discrete-time systems (in contrast to sufficient formulations in [18], [20]), and show that it is the least restrictive in terms of the set of allowable safe inputs, which in turn guarantees optimality when combined with an optimal controller. Further, we propose a more general class of nonlinear control barrier functions for control affine systems with an additional structure that lead to invariance conditions that are affine in the control input and hence, resulting in tractable optimization problems. This enables us to handle the safety optimal control problem for a broader range of applications than the case with linear systems and linear CBFs considered in [18].

Moreover, we derive mixed-integer formulations for basic *Boolean compositions* of multiple CBFs, as an alternative to similar work in [17], [20]–[22], and further provide mixed-integer encodings of secondary Boolean compositions (i.e., *implies*, *exclusive or* and *equivalence*), as well as *if-then-else* statements. These compositions, when combined with the tractable nonlinear CBFs, enable us to guarantee safety for more complicated non-convex or piecewise safe sets and for more general switched systems with non-smooth dynamics using tractable mixed-integer linear/quadratic programs.

Then, equipped with these discrete-time CBF tools, we consider the discrete-time lane keeping problem for autonomous driving that was previously only achieved using a continuous-time formulation [1]. Further, we extend this to the obstacle avoidance problem where a vehicle can avoid an obstacle by choosing to either go around its left or right.

M. Khajenejad, R. Niu and S.Z. Yong are with the School for Engineering of Matter, Transport and Energy, Arizona State University, Tempe, AZ, USA; M. Cavorsi is with the School of Engineering and Applied Sciences, Harvard University, Cambridge, MA, USA; Q. Shen is with the School of Aeronautics and Astronautics, Shanghai Jiao Tong University, Shanghai, P.R. China (email: {mkhajene,rniu6,szyong}@asu.edu, mcavorsi @g.harvard.edu, qiangshen@sjtu.edu.cn). This work is supported in part by NSF grants CMMI-1925110 and CNS-1943545.

* Equal contribution from these authors.

II. Preliminaries and Problem Formulation

A. Notations and Definitions

$\mathbb{R}^n, \mathbb{N}$ and $\mathbb{Z}_n^+$ denote the sets of n-dimensional real numbers, natural numbers and positive integers up to n, respectively. $\mathbf{0}_{m\times n}$ represents the zero matrix in $\mathbb{R}^{m\times n}$.

Definition 1 (SOS-1 Constraint [23])**.** *A special ordered set of degree 1 (SOS-1) constraint*[1] *is a set of scalar variables for which at most one variable in the set may take a value other than zero, denoted as SOS-1: $\{v_1, \ldots, v_N\}$. For instance, if $v_i \neq 0$, then this constraint imposes that $v_j = 0$ for all $j \neq i$.*

Definition 2 (Partition)**.** *A partition of a set/domain $\mathcal{P}$ is a collection of $|\mathcal{J}|$ disjoint subsets $\mathcal{P}_j$ such that $\bigcup_{j\in\mathcal{J}} \mathcal{P}_j = \mathcal{P}$.*

B. Problem Statement

Consider the following class of discrete-time control affine systems with an additional structure:

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} f_1(x_k) \\ f_2(x_k) \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{n_1\times m} \\ g(x_k) \end{bmatrix} u_k \triangleq f(x_k) + \tilde{g}(x_k)u_k, \quad (1)$$

where at time $k \in \mathbb{N}$, $x_k \in \mathbb{R}^n$ and $u_k \in U \subseteq \mathbb{R}^m$ are the state and control input vectors, respectively. We assume that the state vector x_k can be partitioned into two parts as $x_k = \begin{bmatrix} x_{1,k}^\top & x_{2,k}^\top \end{bmatrix}^\top$, where the dynamics of $x_1 \in \mathbb{R}^{n_1}$ ($0 \leq n_1 \leq n$) is autonomously governed by the known vector field $f_1(.) : \mathbb{R}^n \to \mathbb{R}^{n_1}$, i.e., it is not affected by the control input signal u_k, while the dynamics of $x_2 \in \mathbb{R}^{n_2}$ ($0 \leq n_2 \leq n$, $n_1 + n_2 = n$) is governed by the known vector field $f_2(.) : \mathbb{R}^n \to \mathbb{R}^{n_2}$ and is affinely affected by the control input u_k through the function $g : \mathbb{R}^n \to \mathbb{R}^{n_2\times m}$.

Note that the additional structure in (1) is fairly common for systems with higher-order discrete-time nonlinear ARX models and dynamics, including forward Euler discretized mechanical systems with inertia. This structure will also help us to derive nonlinear control barrier functions that lead to tractable constraints that are affine in the control input. The special case with $n_1 = 0$ corresponds to standard control affine systems and the results in this paper still apply with all functions of x_1 interpreted as constant vectors.

Next, we formally define some of the main concepts that will be used to formulate and state our problems of interest. In particular, we consider a (safe) set $\mathcal{S}$ defined as a super level-set of a function $h(\cdot)$:

$$\mathcal{S} \triangleq \{x \in \mathbb{R}^n : h(x) \geq 0\}, \quad (2)$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is any well-defined scalar-valued function, including discontinuous and non-smooth functions, and $\partial\mathcal{S} \triangleq \{x \in \mathbb{R}^n : h(x) = 0\}$ defines the boundary of the set $\mathcal{S}$. Using this definition, we present the notion of controlled invariance of a set $\mathcal{S}$ and introduce a relatively simple formulation for the associated discrete-time control barrier function (DT-CBF), which may also be composed or combined to form more meaningful and expressive (but also more complex) control barrier functions.

[1]Off-the-shelf solvers such as Gurobi [23] can readily handle these constraints, and can significantly reduce the search space for integer variables.

Definition 3. *A set $\mathcal{S}$ is* (forward) controlled invariant *with respect to the system dynamics* (1)*, if for every initial state $x_0 \in \mathcal{S}$, there exists a control input $u_k \in \mathbb{R}^m$ such that the state trajectory always remains in $\mathcal{S}$, i.e., $x_k \in \mathcal{S}$, $\forall k \in \mathbb{Z}$.*

Definition 4 (Discrete-Time Control Barrier Function)**.** *For the discrete-time system (1), the function $h : \mathbb{R}^n \to \mathbb{R}$ is a discrete-time control barrier function (DT-CBF) for the (safe) set $\mathcal{S}$ as defined in* (2)*, if*

$$\exists u \in U \text{ such that } \forall x \in \mathcal{S}, h(f(x) + \tilde{g}(x)u) \geq 0.$$

Equivalently, the function $h : \mathbb{R}^n \to \mathbb{R}$ is a DT-CBF for the (safe) set $\mathcal{S}$ as defined in (2)*, if*

$$\sup_{u\in U} h(f(x) + \tilde{g}(x)u) \geq 0, \qquad \forall x \in \mathcal{S}. \quad (3)$$

Moreover, for any $x \in \mathcal{S}$, we define the corresponding (safe) input set: $K_\mathcal{S}(x) = \{u \in U : h(f(x) + \tilde{g}(x)u) \geq 0\}$.

Now we are ready to specifically state the two problems that this paper seeks to address:

Problem 1 (Synthesis of Tractable Nonlinear Control Barrier Functions)**.** *For the discrete-time system in (1), synthesize a tractable discrete-time control barrier function (DT-CBF) such that a given (safe) set is forward controlled invariant, where the invariance condition, as defined in* (3)*, is affine in the control input (hence, leads to tractable constraints).*

Problem 2 (Compositions of Control Barrier Functions)**.** *Given multiple tractable discrete-time control barrier functions DT-CBFs, find mixed-integer encodings of their (basic and secondary) Boolean compositions, as well as of piecewise control barrier functions, where the invariance condition remains affine in the control input.*

III. Main Results

This section addresses Problems 1 and 2 and in the process, develops tools that enable optimal safety control for autonomous driving in Section IV.

A. Tractable Discrete-Time Control Barrier Functions

This subsection considers the problem of synthesizing tractable discrete-time control barrier functions (DT-CBF) in Problem 1. First, we show that the existence of the DT-CBF, introduced in Definition 4, guarantees the controlled invariance of the (safe) set (2). Then, we propose a class of nonlinear DT-CBFs for systems with the structure in (1) that leads to tractable constraints in optimal control problems.

1) Discrete-Time Control Barrier Functions: Using the proposed DT-CBF in Definition 4 for a (safe) set $\mathcal{S}$ in (2), we now show in the following theorem that its existence is both sufficient and necessary for the (forward) controlled invariance of the set $\mathcal{S}$.

Theorem 1. *Consider the discrete-time system in* (1) *and the (safe) set $\mathcal{S}$ in* (2)*. Then, $\mathcal{S}$ is (forward) controlled invariant if and only if there exists a DT-CBF for $\mathcal{S}$ (cf. Definition 4).*

Proof. With $x_k = x$ and $u_k = u$ for any $x \in \mathcal{S}$ and $u \in K_\mathcal{S}(x)$ at any time step k, the inequality in (3) is satisfied by definition, which means that from (1), we have the following:

$$\sup_{u_k\in U} h(f(x_k)+\tilde{g}(x_k)u_k) = \sup_{u_k\in U} h(x_{k+1}) \geq 0. \quad (4)$$

In other words, $x_k \in \mathcal{S}$ implies that $x_{k+1} \in \mathcal{S}$ with $u_k \in K_{\mathcal{S}}(x_k)$. Further, with the base case of $x_0 \in \mathcal{S}$ (by assumption), we have an inductive proof of sufficiency of the DT-CBF for controlled invariance of $\mathcal{S}$. The necessity can be shown by contraposition. Suppose (3) does not hold. Then, all $u_k \in U$ for some x_k lead to $h(x_{k+1}) < 0$, which means that $\mathcal{S}$ is not controlled invariant. □

Note that our DT-CBF definition is also applicable for discrete-time control affine systems in (1) without the additional structure, i.e., when $n_1 = 0$, and is slightly different from the ones proposed in [18], [20], which have additional terms involving $h(x_k)$ when compared with (4); thus, our definition is arguably more straightforward since it directly imposes the controlled invariance condition without any modifications. More importantly, we can show that the (safe) input set $K_{\mathcal{S}}(x)$ in Definition 4, which is also applicable to control affine systems (with $n_1 = 0$), is a (non-strict) superset of the corresponding input sets based on the definitions in [18], [20], as shown in the following proposition.

Proposition 1. *The (safe) input set $K_{\mathcal{S}}(x)$ for any $x \in \mathcal{S}$ corresponding to the DT-CBF in Definition 4 satisfies $K_{\mathcal{S}}(x) \supseteq K'_{\mathcal{S}}(x)$ and $K_{\mathcal{S}}(x) \supseteq K''_{\mathcal{S}}(x)$, where the input sets $K'_{\mathcal{S}}(x)$ and $K''_{\mathcal{S}}(x)$, defined as $K'_{\mathcal{S}}(x) = \{u \in U : h(f(x) + \tilde{g}(x)u) + (\gamma-1)h(x) \geq 0\}$ and $K''_{\mathcal{S}}(x) = \{u \in U : h(f(x) + \tilde{g}(x)u) + \alpha(h(x)) - h(x) \geq 0\}$, correspond to the definitions of DT-CBF in [18, Proposition 4] and [20, Definition 2], respectively, with $0 \leq \gamma \leq 1$ and a class $\mathcal{K}$ function (i.e., continuous, strictly increasing and $\alpha(0) = 0$) α that satisfies $\alpha(h(x)) < h(x)$.*

Proof. The result follows directly from the observation that

$$u \in K'_{\mathcal{S}}(x) \Rightarrow h(f(x)+\tilde{g}(x)u) \geq (1-\gamma)h(x) \geq 0,$$
$$u \in K''_{\mathcal{S}}(x) \Rightarrow h(f(x)+\tilde{g}(x)u) \geq h(x) - \alpha(h(x)) \geq 0,$$

for all $x \in \mathcal{S}$, with the above choices of γ and α, as well as $h(x) \geq 0$; hence, $u \in K_{\mathcal{S}}$ (cf. Definition 4). □

This means that the DT-CBF definitions in [18], [20] are sufficient for controlled invariance but only necessary with the choice of $\gamma = 1$ and $\alpha(h(x)) = h(x)$. Note that [18] did not provide a necessity proof, while the proof for necessity that is referenced in [20, Theorem 1] is only applicable for $x \in \partial S$, i.e., when $\alpha(h(x)) = h(x) = 0$. Further, the (safe) input set is the least restrictive when using the DT-CBF in Definition 4 and when incorporated into an optimal safety controller, does not lead to sub-optimality. To our understanding, the extra terms in [18], [20] are a legacy from their continuous-time predecessors, e.g., [1, Definition 5], where a relaxation of the invariance condition is introduced to extend the condition for only the boundary of the set $\mathcal{S}$ to the entire domain, including its interior. However, this is not needed for the discrete-time counterpart, because the controlled invariance condition in (3) is already a necessary and sufficient condition for the entire set $\mathcal{S}$.

2) Tractable DT-CBF for Control Affine Systems with Additional Structure: An important consideration when deriving a control barrier function is the tractability of the resulting controlled invariance condition in (3). As observed in [18], unlike the continuous-time counterpart, the controlled invariance condition when incorporated as a constraint in an optimal control problem will in general lead to nonlinear constraints and hence, the authors in [18] focused only on linear systems with linear DT-CBFs. Indeed, this special case is the only one where the controlled invariance condition in (3) is affine in the control input for control affine systems in (1) with $n_1 = 0$.

However, when additional structure is present, i.e., when $n_1 > 0$ for systems with higher-order dynamics, this special case of control affine systems in (1) can also lead to controlled invariance conditions in (3) that are control affine with a careful choice of partially affine DT-CBFs, as follows.

Definition 5 (Partially Affine DT-CBF)**.** *For a discrete-time control affine system with additional structure in* (1)*, the function $h_A : \mathbb{R}^n \to \mathbb{R}$ satisfying*

$$h_A(x) = \mu^\top(x_1)x_2 + \eta(x_1) \quad (5)$$

is a discrete-time partially affine control barrier function (partially affine DT-CBF), which is only affine in x_2, for the (safe) set $\mathcal{S}$ as defined in (2)*, if*

$\exists u \in U$ such that $\forall x \in \mathcal{S}$,

$$h_A(f(x)+\tilde{g}(x)u) = \mu^\top(f_1(x))(f_2(x)+g(x)u) + \eta(f_1(x)) \geq 0,$$

or, equivalently,

$$\sup_{u\in U} h_A(f(x)+\tilde{g}(x)u) \quad (6)$$
$$= \sup_{u\in U} \mu^\top(f_1(x))(f_2(x)+g(x)u) + \eta(f_1(x)) \geq 0, \forall x \in \mathcal{S},$$

where $\mu : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ and $\eta : \mathbb{R}^{n_1} \to \mathbb{R}$ are any nonlinear functions. Moreover, for any $x \in \mathcal{S}$, we define the corresponding (safe) affine input set

$$K^A_{\mathcal{S}}(x) = \{u \in U : \mu^\top(f_1(x))(f_2(x)+g(x)u) + \eta(f_1(x)) \geq 0\}.$$

Remark 1. *The controlled invariance condition in* (6) *is affine in the control input, as desired. Thus, when included as a tractable constraint in an optimal control problem with a quadratic cost, the result is a quadratic program (QP), similar to the continuous-time safety control approach in [1].*

B. Compositions of Multiple and Piecewise DT-CBFs

Next, we provide tools for encoding Boolean compositions of multiple DT-CBFs as well as piecewise/non-smooth CBFs as mixed-integer constraints using SOS-1 constraints (cf. Definition 1) as an alternative to the work in [17], [20]–[22].

First, we consider three basic Boolean operations for composition of multiple control barrier functions $\{h_i(x)\}_{i\in\mathbb{Z}^+_N}$, i.e., $\neg$ (negation), $\wedge$ (conjunction) and $\vee$ (disjunction), where $h_i : \mathbb{R}^n \to \mathbb{R}, \forall i \in \mathbb{Z}^+_N$ are scalar-valued functions. The negation operator is trivial and can be shown by checking if $-h_i(x)$ satisfies the invariance property. Formally, we have

$$\neg(h_i(x) \geq 0) \equiv h_i(x) < 0. \quad (7)$$

As for the disjunction operator $\vee$, we can represent them as

$$\bigvee_{i=1}^{N} h_i(x) \geq 0 \equiv \Big\{\forall i \in \mathbb{Z}_N^+ : h_i(x) \geq s_i, \text{SOS-1} : \{s_i, b_i\}, b_i \in \{0,1\}, \textstyle\sum_{i=1}^{N} b_i \geq 1\Big\}, \quad (8)$$

with s_i being a slack variable, which ensures that there exists at least one $j \in \mathbb{Z}_N^+$ such that $h_j(x) \geq 0$. Moreover, for the conjunction operator $\wedge$, we have

$$\bigwedge_{i=1}^{N} h_i(x) \geq 0 \equiv \Big\{\forall i \in \mathbb{Z}_N^+ : h_i(x) \geq 0\Big\}, \quad (9)$$

which enforces that $h_j(x) \geq 0$ for all $j \in \mathbb{Z}_N^+$.

By leveraging the above three basic Boolean operations, we can further compose the following three secondary Boolean operations found in Boolean algebra:

$$h_i(x) \to h_j(x) \triangleq \neg h_i(x) \vee h_j(x), \quad (10)$$

$$h_i(x) \oplus h_j(x) \triangleq (h_i(x) \vee h_j(x)) \wedge \neg(h_i(x) \wedge h_j(x)), \quad (11)$$

$$h_i(x) \equiv h_j(x) \triangleq \neg(h_i(x) \oplus h_j(x)), \quad (12)$$

which represent the *implication*, *exclusive or* and *equivalence* operations of a pair of control barrier functions $h_i(x)$ and $h_j(x)$, respectively, where we suppressed the ≥ 0 terms in the above for the sake of brevity and clarity.

Finally, we consider the composition of piecewise control barrier functions that enable us to represent more complicated non-convex safe sets, e.g., for the lane keeping problem in Section IV. Specifically, given a partition of the domain $\bigcup_{j \in \mathcal{J}} \mathcal{P}_j$ (cf. Definition 2), where each subregion is represented by the inequality $p_j(x) \leq 0$, the partition/mode-dependent control barrier function can be expressed by an *if-else* statement in the form of '$h_j(x_k) \geq 0$ if $p_j(x) \leq 0$' that can be written using the *implication* operator as

$$p_j(x) \leq 0 \to h_j(x) \Leftrightarrow \neg(p_j(x) \leq 0) \vee h_j(x). \quad (13)$$

Then, with the negation and disjunction operators defined in (7) and (8), we can encode (13) as mixed-integer constraints.

Similar to the discussion above on the tractability of the controlled invariance condition when added as a constraint in an optimal control problem, we will define a piecewise DT-CBF for partially control affine systems in (1) that leads to mixed-integer linear constraints, as follows:

Definition 6 (Piecewise Partially Affine DT-CBF)**.** *Consider a discrete-time control affine system with additional structure in* (1)*. Suppose there exists a partition (cf. Definition 2) of the (safe) set $\mathcal{S}$ in* (2) *with nonlinear mappings $\kappa_j : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ and $\lambda_j : \mathbb{R}^{n_1} \to \mathbb{R}$, $j = 1, \ldots, |\mathcal{J}|$, as follows:*

$$\mathcal{S} = \bigcup_{j \in \mathcal{J}} \{x \triangleq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^n \mid p_j(x) \triangleq \kappa_j(x_1)^\top x_2 + \lambda_j(x_1) \leq 0\},$$

where $\mathcal{J} = \{1, \ldots, |\mathcal{J}|\}$, $|\mathcal{J}|$ is the number of partitions and $p_j(x) \leq 0 \Leftrightarrow p_i(x) > 0, \forall j \neq i \in \mathcal{J}$. Then, the piecewise function $h_P : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$h_P(x) = \begin{cases} \mu_1(x_1)^\top x_2 + \eta_1(x_1), & \text{if } p_1(x) \leq 0, \\ \vdots & \\ \mu_{|\mathcal{J}|}(x_1)^\top x_2 + \eta_{|\mathcal{J}|}(x_1), & \text{if } p_{|\mathcal{J}|}(x) \leq 0, \end{cases} \quad (14)$$

is called a piecewise partially affine DT-CBF for the set $\mathcal{S}$, if $\exists u \in U$ such that $\forall x \in \mathcal{S}, h_P(f(x) + \tilde{g}(x)u) \geq 0$, i.e.,

$$\sup_{u \in U} h_P(f(x) + \tilde{g}(x)u) \geq 0, \ \forall x \in \mathcal{S},$$

and equivalently, for all $j \in \mathcal{J}$,

$$\sup_{u \in U} \mu_j(f_1(x))^\top (f_2(x) + g(x)u) + \eta_j(f_1(x)) \geq 0, \quad \text{if } \kappa_j(f_1(x))^\top (f_2(x) + g(x)u) + \lambda_j(f_1(x)) \leq 0, \quad (15)$$

where $\mu_j : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ and $\eta_j : \mathbb{R}^{n_1} \to \mathbb{R}$ can be any nonlinear mappings. Moreover, we define the corresponding (safe) piecewise affine input set

$$K_{\mathcal{S}}^P(x) = \{u \in U : \mu_j(f_1(x))^\top (f_2(x) + g(x)u) + \eta_j(f_1(x)) \geq 0 \ \text{if } \kappa_j(f_1(x))^\top (f_2(x) + g(x)u) + \lambda_j(f_1(x)) \leq 0, \ \forall j \in \mathcal{J}\}. \quad (16)$$

Remark 2. *It can be observed that the controlled invariance condition in* (15) *is piecewise control affine. Hence, when incorporated as a constraint in an optimal control formulation with a linear/quadratic cost, the result is a mixed-integer linear/quadratic program (MILP/MIQP). Similar results can also be derived in a straightforward manner when the system dynamics are switched among a set of control affine dynamics with additional structure, and thus, a detailed description is omitted for the sake of brevity.*

Moreover, since Theorem 1 and Proposition 1 hold for (safe) sets $\mathcal{S}$ with well-defined functions, including piecewise functions (cf. (2)), the same results also apply to the piecewise partially affine DT-CBF defined in Definition 6.

IV. Application to Driving Safety Problems

A. Lane Keeping

The goal of the Lane Keeping (LK) problem is to keep a vehicle in the middle of a desired lane that may be curved by controlling the vehicle's lateral displacement. The simulation example conveyed in this work was largely inspired by the LK example in [1], where the authors developed a continuous-time CBF-based approach to solve this problem. By contrast, we consider the development of a discrete-time CBF approach and show that the resulting optimal control problem is now a mixed-integer quadratic program (MIQP) or, equivalently, two *parallel* quadratic programs (QPs), as opposed to a single QP for continuous-time systems in [1]. Nonetheless, this discrete-time implementation is useful for implementation in digital microprocessors/controllers.

Similar to [1], we consider the vehicle model in [24] (with forward Euler time-discretization with sampling time t_s):

$$x'_{k+1} = (I + At_s)x'_k + Bt_s u_k + Et_s r_{d,k}. \quad (17)$$

where

$$A = \begin{bmatrix} 0 & 1 & V_0 & 0 \\ 0 & -\frac{C_f + C_r}{MV_0} & 0 & \frac{bC_r - aC_f}{MV_0} - V_0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{bC_r - aC_f}{I_z V_0} & 0 & -\frac{a^2 C_f + b^2 C_r}{I_z V_0} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{C_f}{M} \\ 0 \\ a\frac{C_f}{I_z} \end{bmatrix}, E = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

The states $x'_k \triangleq \begin{bmatrix} y_k & \nu_k & \psi_k & r_k \end{bmatrix}^\top$ are the lateral displacement of the car from the center of the lane (y_k), the car's lateral velocity (ν_k), the yaw angle of the car with respect to the lane center (ψ_k) and the yaw rate of the car (r_k).

The input u_k is the angle of the front tires at the current time k. Road curvature is a *known* disturbance to the system and the road curves at a rate of $r_{d,k} = \frac{V_0}{R_k}$, where V_0 is the longitudinal velocity of the vehicle and R_k is the radius of curvature of the road at time step k. The parameters M, I_z, a, b, C_f and C_r are the vehicle mass, moment of inertia about the center of mass, the distance from the center of mass to the front and rear tires and tire parameters, respectively. Further, we define $c_f \triangleq \frac{C_f}{M}$ and $c_r \triangleq \frac{C_r}{M}$.

First, we put the system (17) into the control affine with additional structure form in (1) with a reduced state $x_k \triangleq \begin{bmatrix} x_{1,k} & x_{2,k} \end{bmatrix}^\top = \begin{bmatrix} y_k & \nu_k \end{bmatrix}^\top$, where ψ_k and r_k are known/measured parameters, $g(x_k) = t_s c_f$, $f_1(x_k) = \begin{bmatrix} 1 & t_s \end{bmatrix} x_k + t_s V_0 \psi_k$ and $f_2(x_k) = \begin{bmatrix} 1 & -t_s \frac{c_f + c_r}{V_0} \end{bmatrix} x_k + t_s \frac{bc_r - ac_f}{V_0} r_k$. Next, we consider two LK constraints:

1) ***Acceleration Constraint****:* The first constraint is to prevent unbounded lateral acceleration a_k of the car:

$$|a_k| = \tfrac{1}{t_s} |(v_{k+1} - v_k)| \leq a_{max}, \ \forall k \in \mathbb{N}, \tag{18}$$

where $v_k = \frac{1}{t_s}(y_{k+1} - y_k)$ is the instantaneous lateral velocity. From (17), we have $\frac{1}{t_s}(v_{k+1} - v_k) = f_0 + c_f u_k$, where $F_0 \triangleq C_f \frac{\nu_k + a r_k}{V_0} + C_r \frac{\nu_k - b r_k}{V_0} + M V_0 r_{d,k}$ and $f_0 \triangleq \frac{F_0}{M}$. So, the constraint (18) can then be written as

$$\begin{bmatrix} 1 & -1 \end{bmatrix}^\top u_k \leq \tfrac{1}{C_f} \begin{bmatrix} (M a_{max} + F_0) & (M a_{max} - F_0) \end{bmatrix}^\top. \tag{AC}$$

2) ***Lane Centering Constraint****:* This second constraint keeps the car from drifting too far away from the middle of the lane. This can be done by restricting the maximum lateral displacement:

$$|y_k| \leq y_{max}, \ \forall k \in \mathbb{N}. \tag{19}$$

As described in [1], a typical United States lane is 12 feet wide while a car is about 6 feet wide, so the maximum lateral displacement the car can safely experience is 3 feet to either side, so $y_{max} = 3$ feet ≈ 0.9 meters.

The next proposition proposes a piecewise partially affine DT-CBF that can enforce the controlled invariance of the lane centering constraint as a safe set, i.e., $\mathcal{S}_{LK} = \{x \in \mathbb{R}^2 : (19) \text{ holds}\}$, subject to the acceleration input constraint, i.e., $U = \{u \in \mathbb{R} : (18) \text{ holds}\}$.

Lemma 1. *The function* $h_{LK} : \mathbb{R}^2 \to \mathbb{R}$

$$h_{LK}(x) = \sqrt{2a_{max}(y_{max} - \operatorname{sgn}(v) y) + \frac{1}{4} a_{max}^2 t_s^2} - v', \tag{20}$$

where $x \triangleq \begin{bmatrix} y & \nu \end{bmatrix}^\top$, $v \triangleq \nu + V_0 \psi$ *is the instantaneous lateral velocity and* $v' \triangleq |v| + \frac{1}{2} a_{max} t_s$*, and equivalently,*

$$h_{LK}(x) = \begin{cases} \sqrt{2a_{max}(y_{max} - x_1) + \frac{1}{4} a_{max}^2 t_s^2}, & \text{if} -x_2 - V_0 \psi \leq 0, \\ \sqrt{2a_{max}(y_{max} + x_1) + \frac{1}{4} a_{max}^2 t_s^2}), & \text{if} \ \ x_2 + V_0 \psi \leq 0, \end{cases} \tag{21}$$

with $x_1 = y$ *and* $x_2 = \nu$*, is a valid piecewise partially affine DT-CBF*[2] *in the form of* (14) *for the (safe) set* $\mathcal{S}_{LK} = \{x \in \mathbb{R}^2 : (19) \text{ holds}\}$*. Moreover, the corresponding (safe) piecewise affine input set* $K_{\mathcal{S}_{LK}}(x)$ *in the form of* (16) *(cf. Definition 6) for any* $x_k = \begin{bmatrix} y_k & \nu_k \end{bmatrix}^\top \in \mathcal{S}$ *is given by the set of inputs* $u_k \in U$ *that satisfy*

$$\begin{aligned} \eta_k^+ - z_k - t_s c_f u_k \geq 0, & \quad \text{if } z_k + t_s c_f u_k \geq 0, \\ \eta_k^- + z_k + t_s c_f u_k \geq 0, & \quad \text{if } z_k + t_s c_f u_k < 0, \end{aligned} \tag{22}$$

where $\eta_k^\pm \triangleq \sqrt{2a_{max}(y_{max} \mp y_{k+1}) + \frac{1}{4} a_{max}^2 t_s^2} - \frac{1}{2} a_{max} t_s$, $z_k \triangleq V_0 \psi_{k+1} + (1 + t_s \alpha) \nu_k + t_s \beta r_k$, $\alpha = -\frac{1}{V_0}(c_f + c_r)$, $\beta = \frac{1}{V_0}(bc_r - ac_f) - V_0$, $\psi_{k+1} = \psi_k + t_s(r_k - r_{d,k})$ *and* $y_{k+1} = y_k + t_s(\nu_k - V_0 \psi_k)$*, which can be implemented with SOS-1 constraints (cf. Definition 1), as follows:*

$$\begin{aligned} &\eta_k^+ - z_k - t_s c_f u_k + s_1 \geq 0, \ z_k + t_s c_f u_k + s_1 \geq 0, \\ &\eta_k^- + z_k + t_s c_f u_k + s_2 \geq 0, \ -z_k - t_s c_f u_k + s_2 > 0, \\ &\textit{SOS-1} : \{s_1, s_2\}, \ s_1, s_2 \geq 0, \end{aligned} \tag{LC-CBF}$$

which are mixed-integer linear constraints in u_k*.*

Proof. First, we construct the safe set $\mathcal{S}$ by showing that $h_{LK}(x) \geq 0$ is equivalent to (19). For any (initial) displacement y and instantaneous velocity v, with the maximum allowable acceleration/deceleration given $a = -\operatorname{sgn}(v) a_{max}$ (cf. (18)) it takes time $T = \frac{|v|}{a_{max} t_s}$ to reach $v_T = 0$. Correspondingly, the furthest lateral displacement with maximum acceleration/deceleration to come to a full stop is given by $y_T = y + \frac{1}{2a_{max}} v|v| + \frac{1}{2} v t_s$. Taking the travel direction into consideration using $\operatorname{sgn}(v)$, we can then impose the lane centering constraint in (19) as: $\operatorname{sgn}(v) y_T = \operatorname{sgn}(v) y + \frac{1}{2a_{max}} v^2 + \frac{1}{2}|v| t_s \leq y_{max} \Leftrightarrow v^2 + |v| a_{max} t_s \leq 2a_{max}(y_{max} - \operatorname{sgn}(v) y)$. Completing the square yields $(|v| + \frac{1}{2} a_{max} t_s)^2 - \frac{1}{4} a_{max}^2 t_s^2 \leq 2a_{max}(y_{max} - \operatorname{sgn}(v) y)$, and considering its square root leads to our choice of $h_{LK}(x)$ in (20). Intuitively, this $h_{LK}(x) \geq 0$ ensures that for any state x, there is enough time in the future to come to a complete stop before reaching the lane boundary. Since the system states are continuous, this includes the case that the lateral displacement at the next time step starting at y with velocity v does not violate the lane centering constraint; thus, the controlled invariance condition in (3) holds and h_{LK} is a DT-CBF for $\mathcal{S}_{LK}$. Moreover, this DT-CBF in (20) can be easily shown to be equivalent to the piecewise partially affine DT-CBF in (21) by considering the two cases for $\operatorname{sgn}(v) = \operatorname{sgn}(\nu + V_0 \psi)$.

Next, we show that $K_{\mathcal{S}_{LK}}$ can be expressed as mixed-integer linear constraints using the composition tools for piecewise functions (as discussed in Remark 2). Now, for $x_k = \begin{bmatrix} y_k & \nu_k \end{bmatrix}^\top$ and $v_k = \frac{1}{t_s}(y_{k+1} - y_k)$ with $y_{k+1} = y_k + t_s(\nu_k + V_0 \psi_k)$ and the definition $\eta_k^\pm \triangleq \sqrt{2a_{max}(y_{max} \mp y_{k+1}) + \frac{1}{4} a_{max}^2 t_s^2} - \frac{1}{2} a_{max} t_s$, the controlled invariance condition $h_{LK}(x_{k+1}) \geq 0$ can be written as a piecewise condition:

$$\begin{aligned} \eta_k^+ + \tfrac{y_{k+1}}{t_s} - \tfrac{y_{k+2}}{t_s} \geq 0, & \ \text{if } v_{k+1} \geq 0, \\ \eta_k^- - \tfrac{y_{k+1}}{t_s} + \tfrac{y_{k+2}}{t_s} \geq 0, & \ \text{if } v_{k+1} < 0, \end{aligned} \tag{23}$$

where $y_{k+2} = y_{k+1} + t_s z_k + t_s^2 c_f u_k$, $z_k \triangleq V_0 \psi_{k+1} + (1 + t_s \alpha) \nu_k + t_s \beta r_k$, $\alpha = -\frac{1}{V_0}(c_f + c_r)$, $\beta = \frac{1}{V_0}(bc_r - ac_f) - V_0$ and $\psi_{k+1} = \psi_k + t_s(r_k - r_{d,k})$. Then, using (13), the piecewise condition in (23) is equivalent to (LC-CBF). $\square$

[2] Note that in the limit when the sampling time t_s tends to zero, $h_{LK}^2(x)$ (from (20)) becomes the continuous-time CBF in [1, Eq. (53)].

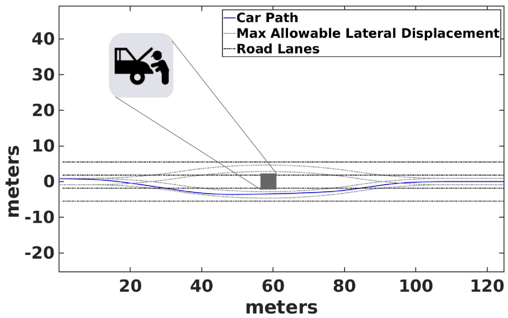

Fig. 1: The vehicle must go around the obstacle to the left or to the right. This is represented by two new lanes to follow. The vehicle in this situation chooses to follow the right lane.

Next, we adopt the optimal control framework with a quadratic cost and a legacy controller in [1] to select the optimal input from the (safe) input set $K_{\mathcal{S}_{LK}}$, as follows:

Mixed-Integer Quadratic Program for LK: The DT-CBF is combined with a linear feedback controller $u_k = -K(x_k - x_{ff,k})$, where K is a (legacy) controller gain and $x_{ff,k} = \begin{bmatrix} 0 & r_{d,k} \end{bmatrix}^\top$, as well as the acceleration and lane centering constraints, (18) and (19), respectively, resulting in the following mixed-integer quadratic program MIQP):

$$\mathbf{u}_k^* = \underset{\mathbf{u}_k=[u_k,\delta]^\top}{\text{argmin}} \quad \tfrac{1}{2}\mathbf{u}_k^\top H \mathbf{u}_k + F^\top \mathbf{u}_k \tag{24}$$
$$\text{s.t. (AC), (LC-CBF) hold and } u_k{=}-K(x_k{-}x_{ff,k}){+}\delta,$$

where δ is a relaxation variable such that the linear feedback controller forms a soft constraint that is only achieved if the required (safety) constraints are not violated, $H \in \mathbb{R}^{2\times 2}$ is positive definite, and $F \in \mathbb{R}^2$. Note that if a mixed-integer program is undesired, the controller in (24) can be equivalently achieved with two *parallel* quadratic programs with (LC-CBF) replaced by each of the conditions in (23), where the smaller feasible solution of the two chosen as $\mathbf{u}_k^*$.

B. Obstacle Avoidance

Next, we consider the Obstacle Avoidance (OA) problem as an extension to the LK problem, where in the event that there is an obstacle in the road lane, the vehicle avoiding the obstacle to the left or right (cf. Figure 1) can be modeled by an LK problem in which the lane splits into two lanes going around the obstacle on either side, one with a curve rate of $r_{d_1,k}$ and another with a curve rate of $r_{d_2,k}$. Obviously, the vehicle cannot remain in both lanes as they split around the obstacle and we encode the choice between the left and right lanes using a conjunction ('OR' or $\vee$) of two barrier functions for each lane, i.e., with $(h_{LK,l} \geq 0) \vee (h_{LK,r} \geq 0)$.

Mixed-Integer Quadratic Program for OA: When incorporated into an optimal control framework as in (24), we obtain another mixed-integer quadratic program by virtue of the composition tools we developed in Section III-B:

$$\begin{aligned} \mathbf{u}_k^* = \quad & \underset{\mathbf{u}_k=[u_k,\delta]^\top}{\text{argmin}} \quad \tfrac{1}{2}\mathbf{u}_k^\top H \mathbf{u}_k + F^\top \mathbf{u}_k \\ \text{s.t.} \quad & ((\text{AC}_l) \wedge (\text{LC-CBF}_l)) \vee ((\text{AC}_r) \wedge (\text{LC-CBF}_r)), \\ & u_k = -K(x_k - x_{ff,k}) + \delta, \end{aligned} \tag{25}$$

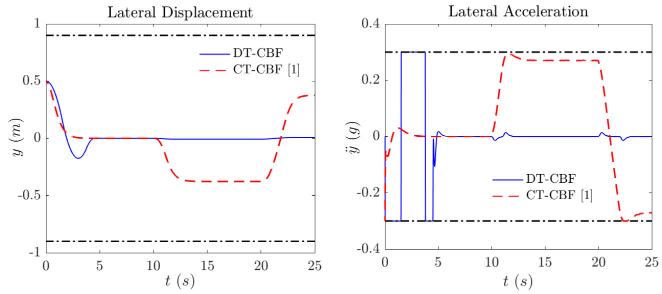

Fig. 2: Comparison between the proposed DT-CBF approach (blue solid lines) and the CT-CBF in [1] (red dashed lines).

where (AC_l) and (LC-CBF_l) are (AC) and (LC-CBF) based on $r_{d_1,k}$, respectively, while (AC_r) and (LC-CBF_r) are based on $r_{d_2,k}$. Similarly, the controller (25) can be equivalently designed using four *parallel* quadratic programs, where the smallest feasible solution is chosen as $\mathbf{u}_k^*$.

C. Simulation Results

We consider the following parameter values in the simulations of both the LK and OA problems: $V_0 = 8.33$ m/s, $C_f = 133000$ N/rad, $C_r = 98800$ N/rad, $M = 1650$ kg, $a = 1.11$ m, $b = 1.59$ m, $I_z = 2315.3$ m^{2}kg, $g = 9.81$ m/s^2, $a_{max} = 0.3g$ and $t_s = 0.01$ s. The feedback gain K was determined to place the poles at $\{0.95, 0.8, 0.85, 0.9\}$ using MATLAB's `place` command. All simulations were implemented in MATLAB 2020a with Gurobi v9.1.1 [23] on a 2.6GHz Intel Core i7-10750H CPU with 16GB RAM.

1) LK Problem: First, we demonstrate and compare the effectiveness of our DT-CBF approach for the LK problem with the continuous-time approach in [1] (CT-CBF). As shown in Figure 2, with the initial state set to $x_0 = \begin{bmatrix} 0.5 & 0 & 0 & 0 \end{bmatrix}^\top$, the lateral displacement and acceleration for both DT-CBF and CT-CBF stay within the desired bounds of ± 0.9 m and $\pm 0.3g$, respectively, but their behaviors are rather different. The lateral acceleration with the DT-CBF is more "aggressive," but the lateral displacement remains much closer to zero, meaning the vehicle stays closer to the center of the lane, as desired. On the other hand, the vehicle drifts up to approximately 0.4 meters from the center of the lane once the road starts to curve at $t = 10$ seconds with the CT-CBF. Further, since the control input is roughly proportional to the lateral acceleration, it appears that smaller inputs are needed in the long run when using the DT-CBF.

In terms of the computation times of the optimization problems corresponding to the CT-CBF in [1] (with a single QP), the DT-CBF (i.e., (24) with an MIQP) and the DT-CBF based on solving two parallel QPs (with (LC-CBF) replaced by each of the piecewise conditions in (23)), the average elapsed times were 0.0320, 0.0360 and 0.0522 seconds, respectively. As expected, solving a single QP when using CT-CBF is the fastest but its performance is dependent on the choice of its class $\mathcal{K}$ function α, while the performance of DT-CBF is independent of the choice of α but at the cost of slightly more computation.

2) OA Problem: An example scenario for the obstacle avoidance problem is while driving down a road and noticing

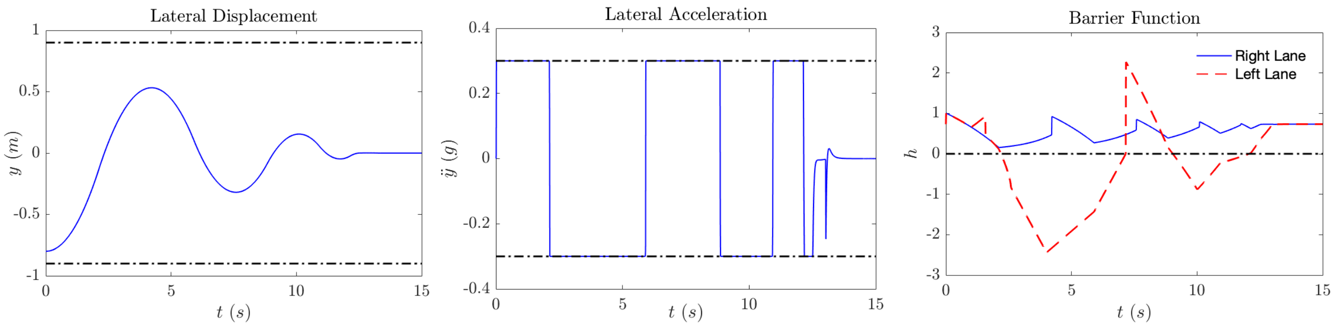

Fig. 3: The lateral displacement (left) of the car is bounded by 0.9 meters from the chosen lane. The lateral acceleration (middle) is bounded by $0.3g$. The barrier function (right) that was chosen in the conjunction ('OR') condition stays positive throughout the simulation (blue solid line), while the other does not (red dashed line).

an obstacle up ahead where the vehicle either needs to go around the obstacle to the left, or to the right. As opposed to a vehicle following a curved road and staying within a safe distance of the road center, the road is simulated to curve in two opposite directions $r_{d_1,k}$ and $r_{d_2,k} = -r_{d_1,k}$ and the vehicle can choose whether to avoid the obstacle by driving around it to the left or right (assuming no other obstacles).

To simulate this OA problem, we implemented the mixed-integer quadratic program in (25) with the initial condition set to $x_0 = \begin{bmatrix}-0.8 & 0 & 0 & 0\end{bmatrix}^\top$, and the results are shown in Figure 3, where the lateral displacement and lateral acceleration remained within the desired constraints, as expected. Moreover, for the chosen lane (to the right in this case), the control barrier function h for that lane (cf. Figure 3, right, blue solid line) remained positive, but not for the other barrier function (red dashed line). Moreover, from running several simulations, it appears that the vehicle decides the lane based on the lateral direction it is already accelerating in.

V. Conclusion

A novel formulation for control barrier functions for ensuring the safety of discrete-time systems was presented and was shown to be necessary and sufficient for controlled invariance and less restrictive than existing formulations. In addition, we proposed nonlinear DT-CBFs for partially control affine systems, whose controlled invariance conditions are affine in the control input, which means that they can be included as tractable affine constraints in safety optimal control problems for a broader range of applications and safety conditions than the state-of-the-art. Furthermore, we derived mixed-integer formulations for Boolean compositions of multiple CBFs as well as for piecewise CBFs. Finally, these new sets of DT-CBF tools were applied and tested in simulations for lane keeping and obstacle avoidance in driving safety.

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