Optimization-Based Approaches for Affine Abstraction and Model Discrimination of Uncertain Nonlinear Systems

Zeyuan Jin, Qiang Shen and Sze Zheng Yong

Abstract— **This paper presents novel optimization-based approaches for affine abstraction and model discrimination of uncertain nonlinear systems in the form of nonlinear (basis) functions with uncertain coefficients. First, we propose a mesh-based affine abstraction method to conservatively approximate the uncertain nonlinear functions in the sense of the inclusion of all possible trajectories by two affine hyperplanes in each bounded subregion of the state space. As the affine abstraction is an over-approximation of the original system, any model invalidation guarantees for the abstraction also hold for the original system. Next, we extend existing methods to solve the (passive) model discrimination problem for the piecewise affine interval models obtained from abstraction by leveraging model invalidation. It is shown that the model invalidation and discrimination problems can be recast as the feasibility of a mixed-integer linear program (MILP). Finally, the efficiency of the approach is illustrated with numerical examples motivated by intent/formation identification of autonomous swarm systems.**

I. Introduction

In recent years, there is a growing interest in abstraction-based methods on analyzing reachability, estimating state and synthesizing controller for cyber-physical systems (CPS). Since CPS are integrations of networks and embedded computers with physical processes, they often have complex (uncertain, nonlinear or hybrid) dynamics, which makes the controller and estimator design challenging. To overcome this, abstraction approaches that conservatively approximate the original complex dynamics with simpler dynamics have been developed [1]. These abstracted simpler systems enable us to apply the well-developed controller or observer design methods and ensure that guarantees for the simpler systems also hold for the original systems [2]–[4].

Literature Review. In general, *abstraction* is a process that approximates the system dynamics by simpler models that "include" all possible trajectories of the original system. Methods for abstraction have been proposed for several types of systems, such as linear systems [5], nonlinear systems [6], and discrete-time hybrid systems [7]. In [3], nonlinear dynamics was over-approximated as a linear affine system with a bounded disturbance accounting for the abstraction error and ensuring conservativeness. In [8], Singh *et. al* proposed a mesh-based affine abstraction approach for nonlinear systems with different degrees of smoothness, where a pair of piecewise affine functions brackets/encloses the original dynamics in each subregion with a given approximation accuracy. In [9], two affine hyperplanes were constructed to conservatively approximate uncertain affine discrete-time systems, in which system matrices were assumed to be uncertain and represented by interval matrices/vectors. However, all above mentioned methods are only applicable for known nonlinear or uncertain affine models, and not for uncertain nonlinear models that we consider in this work.

On the other hand, passive model discrimination aims to distinguish/separate models by exploiting the measured input-output data and a priori information of the system (e.g., [10], [11]). This is typically achieved using model invalidation, which aims to determine whether a finite sequence of experimental input-output data measured from a system can be generated by one member in an admissible model set [12]. Recently, various model invalidation methods have been developed for linear parameter varying systems [13], [14], nonlinear systems [15], switched auto-regressive models [16], [17], and switched affine systems [11], [18]. To the best of our knowledge, the model invalidation results for uncertain nonlinear systems are not available in the literature, with the main difficulty being the nonlinearities and uncertainties in the system.

Contributions. In this paper, we propose optimization-based methods to address the affine abstraction problem for a class of uncertain nonlinear systems and the corresponding model discrimination problem based on the resulting abstracted piecewise affine interval models. Specifically, we consider the class of uncertain nonlinear dynamics consisting of nonlinear basis functions with uncertain coefficients/parameters that are represented by interval matrices. We first develop a mesh-based affine abstraction approach to over-approximate the uncertain nonlinear systems by two hyperplanes in each subregion such that all (worst-case) system behaviors of the original system are included by the abstraction. In particular, any model discrimination and invalidation guarantees for the abstraction also hold for the original models. Leveraging linear interpolation and properties of interval matrices, we solve a linear program (LP) to obtain affine abstraction, which over-approximates the uncertain nonlinear systems as piecewise affine abstractions. Then, we further propose an approach to solve the model discrimination problem for piecewise affine abstractions, by recasting it as the feasibility of a mixed-integer linear program (MILP), for which off-the-shelf solvers are readily available. When compared with our previous efforts [8], [9], we take advantage of both papers to enable the proposed affine abstraction to over-approximate the uncertain nonlinear systems. Moreover, as opposed to switched affine systems in [11], [18], we consider model invalidation for piecewise

The authors are with School for Engineering of Matter, Transport and Energy, Arizona State University, Tempe, USA (email: {zjin43,qiang.shen,szyong}@asu.edu)

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affine abstractions, which represent a more general class of systems. Finally, we demonstrate the effectiveness of our affine abstraction-based model discrimination approach for intent/formation estimation of a swarm of vehicles.

II. Background

A. Notation

For a vector $v \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{p\times q}$, $\|v\|_i$ and $\|M\|_i$ denote their (induced) i-norm with $i = \{1, 2, \infty\}$. $[n]$ is an initial segment $1, 2, \ldots, n$ of the natural numbers. An interval matrix $\mathcal{M}$ is defined as a set of matrices of the form $\mathcal{M} = [M_l, M_u] = \{M \in \mathbb{R}^{p\times q} : M_l \leq M \leq M_u\}$, where M_l and M_u are $p \times q$ matrices, and the inequality is to be understood componentwise.

B. Modeling Frameworks

Consider a class of uncertain nonlinear discrete-time system model $\mathcal{G}$:

$$\begin{aligned} x_{k+1} &= A_k\phi(x_k, u_k) + w_k, \\ y_k &= C_k x_k + v_k, \end{aligned} \tag{1}$$

where the nonlinear function $A_k\phi(x_k, u_k)$ is a linear combination of nonlinear basis functions $\phi(x_k, u_k)$ with uncertain coefficients/parameters denoted by $A_k \in \mathcal{A}$ with bounded sets $\mathcal{A} = [A_l, A_u] \subset \mathbb{R}^{n\times d}$, $x_k \in \mathcal{X}$ denotes system state at time instant k with a bounded set $\mathcal{X} = [X_l, X_u] \subset \mathbb{R}^n$, $u_k \in \mathcal{U}$ denotes control input with bounded set $\mathcal{U} = [U_l, U_u] \subset \mathbb{R}^m$ and y denotes the system output at time instant k, w_k and v_k are the bounded process noise and measurement noise satisfying $\|w_k\| \leq \varepsilon_w$ and $\|v_k\| \leq \varepsilon_v$, respectively. The nonlinear basis function $\phi : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^d$ is the vector field describing the nonlinear dynamics of the system. We assume ϕ is Lipschitz continuous. If a feedback control law is used in the system, the closed-loop dynamics can also be considered.

Further, we define a *partition* of the compact state-input domain $\mathcal{X} \times \mathcal{U} \subseteq \mathbb{R}^{n+m}$ as follows:

Definition 1 (Partition)**.** *A partition $\mathcal{I}$ of the closed bounded region $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^{n+m}$ is a collection of p subregions $\mathcal{I} = \{I_i | i \in [p]\}$ such that $\mathcal{X} \times \mathcal{U} \subseteq \bigcup_{i=1}^p I_i$ and $I_i \cap I_j = \partial I_i \cap \partial I_j$, $\forall i \neq j \in [p]$, where ∂I_ℓ is the boundary of set I_ℓ.*

For each subregion $I_i \in \mathcal{I}$ that partitions the domain of interest, we aim to over-approximate/abstract the nonlinear f by a pair of affine functions $\underline{f}_i$ and $\overline{f}_i$ such that for all $(x_k, u_k) \in I_i$, the function $f(x_k, u_k)$ is sandwiched by the pair of affine functions, i.e., $\underline{f}_i(x_k, u_k) \leq f(x_k, u_k) \leq \overline{f}_i(x_k, u_k)$. These affine functions with respect to f over $I_i \in \mathcal{I}$ are chosen as

$$\underline{f}_i(x_k, u_k) = \underline{A}_i x_k + \underline{B}_i u_k + \underline{h}_i, \tag{2}$$

$$\overline{f}_i(x_k, u_k) = \overline{A}_i x_k + \overline{B}_i u_k + \overline{h}_i, \tag{3}$$

where the matrices $\underline{A}_i$, $\overline{A}_i$, $\underline{B}_i$, $\overline{B}_i$, and the vectors $\underline{h}_i$ and $\overline{h}_i$ are constant and of appropriate dimensions. Let $(\underline{\mathcal{F}}, \overline{\mathcal{F}})$ be a pair of families of affine functions with $\underline{\mathcal{F}} = \{\underline{f}_1, \ldots, \underline{f}_p\}$ and $\overline{\mathcal{F}} = \{\overline{f}_1, \ldots, \overline{f}_p\}$. Then, the nonlinear function $f : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^d$ is over-approximated with a pair of affine families $(\underline{\mathcal{F}}, \overline{\mathcal{F}})$ over a partition $\mathcal{I}$ (i.e., a pair of piecewise affine functions) if $\underline{f}_i(x_k, u_k) \leq f(x_k, u_k) \leq \overline{f}_i(x_k, u_k)$, $\forall i \in [p]$ and $\forall (x_k, u_k) \in I_i$.

The abstracted piecewise affine interval model $\mathcal{H}$ is then:

$$\begin{aligned} \begin{pmatrix} \underline{A}_i x_k + \underline{B}_i u_k \\ +\underline{h}_i + w_k \end{pmatrix} \leq x_{k+1} \leq \begin{pmatrix} \overline{A}_i x_k + \overline{B}_i u_k \\ +\overline{h}_i + w_k \end{pmatrix}, \forall i \in [p], \\ y_k = C_k x_k + v_k. \end{aligned} \tag{4}$$

Moreover, we quantify the *quality* of our affine abstraction based on the following definition of approximation error.

Definition 2 (Approximation Error [8])**.** *Consider a partition $\mathcal{I} = \{I_i | i \in [p]\}$ of $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^{m+n}$. If a pair of affine families $(\underline{\mathcal{F}}, \overline{\mathcal{F}})$ over-approximate a nonlinear function f over the partition $\mathcal{I}$, then the approximation error with respect to the nonlinear dynamics is defined as $e(\underline{\mathcal{F}}, \overline{\mathcal{F}}) = \max_{i\in[p]} \max_{(x_k,u_k)\in I_i} \|\overline{f}_i(x_k, u_k) - \underline{f}_i(x_k, u_k)\|_\infty$.*

Next, to solve the model discrimination problem via model invalidation, we further adopt the definition in [11] of the length-N behavior of original uncertain nonlinear and abstracted piecewise affine interval models, $\mathcal{G}$ and $\mathcal{H}$:

Definition 3 (Length-N Behavior of Original Model $\mathcal{G}$)**.** *The length-N behavior of the uncertain nonlinear model $\mathcal{G}$ is the set of all length-N input-output trajectories compatible with $\mathcal{G}$, given by the set*

$$\begin{aligned} &\mathcal{B}^N(\mathcal{G}) := \{\{u_k, y_k\}_{k=0}^{N-1} \mid u_k \in \mathcal{U} \text{ and } \exists x_k \in \mathcal{X}, \\ &w_k \in \mathcal{W}, v_k \in \mathcal{V}, \text{ for } k \in \mathbb{Z}_{N-1}^1, \text{ s.t. (1) holds}\}. \end{aligned} \tag{5}$$

Definition 4 (Length-N Behavior of Abstracted Model $\mathcal{H}$)**.** *The length-N behavior of the abstracted piecewise affine interval model $\mathcal{H}$ is the set of all length-N input-output trajectories compatible with $\mathcal{H}$, given by the set*

$$\begin{aligned} &\mathcal{B}^N(\mathcal{H}) := \{\{u_k, y_k\}_{k=0}^{N-1} \mid \exists (x_k, u_k) \in I_i, i \in [p], \\ &w_k \in \mathcal{W}, v_k \in \mathcal{V}, \text{ for } k \in \mathbb{Z}_{N-1}^1, \text{ s.t. (4) holds}\}. \end{aligned} \tag{6}$$

Using the above definitions of system behaviors as well as the fact that $\mathcal{H}$ is an affine abstraction of $\mathcal{G}$ (by construction), we can conclude that $\mathcal{B}^N(\mathcal{G}) \subseteq \mathcal{B}^N(\mathcal{H})$.

III. Problem Statement

We now formulate the problems of interest to this paper.

Problem 1 (Affine Abstraction)**.** *For a given nonlinear n-dimensional vector field $f(x_k, u_k) = A_k\phi(x_k, u_k)$ with $(x_k, u_k) \in \mathcal{X} \times \mathcal{U}$ and a given desired accuracy ε_f, find a partition $\mathcal{I} = \{I_1, \ldots, I_p\}$ and a pair of n-dimensional family of affine hyperplanes $\overline{\mathcal{F}} = \{\overline{f}_1, \ldots, \overline{f}_p\}$ and $\underline{\mathcal{F}} = \{\underline{f}_1, \ldots, \underline{f}_p\}$ such that:*

$$\begin{aligned} &e(\underline{\mathcal{F}}, \overline{\mathcal{F}}) \leq \varepsilon_f, \\ &\underline{f}_i(x_k, u_k) \leq A_k\phi(x_k, u_k) \leq \overline{f}_i(x_k, u_k), \\ &\forall (x_k, u_k) \in \mathcal{I}_i, \forall i \in [p], \forall A_k \in \mathcal{A}, \end{aligned} \tag{7}$$

where $e(\underline{\mathcal{F}}, \overline{\mathcal{F}})$ is the approximation error (cf. Definition 2). The pair of affine families $(\underline{\mathcal{F}}, \overline{\mathcal{F}})$ is then the abstracted piecewise affine interval model (i.e., affine abstraction of the nonlinear uncertain dynamics).

Problem 2 (Model Discrimination amongst $\{\mathcal{G}_\ell\}_{\ell=1}^{N_m}$). *Given a sequence of input-output trajectory $\{u_k, y_k\}_{k=0}^{N-1}$, N_m uncertain nonlinear models $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{N_m}$ and an integer N, determine which model the trajectory belongs to. That is, to find an i that satisfies*

$$\mathcal{B}^N(\mathcal{G}_i) \neq \emptyset \wedge (\mathcal{B}^N(\mathcal{G}_j) = \emptyset, \forall j \in \mathbb{Z}_{N_m}^1, j \neq i). \quad (8)$$

However, since the original models $\mathcal{G}_\ell$ are uncertain and nonlinear, Problem 2 is non-trivial to solve directly. Hence, we aim to address Problem 2 using a related problem that, if solved, also provides a solution to Problem 2. Specifically, we plan to consider a two-step process, where the first step consists of solving Problem 1 to obtain the over-approximation of the uncertain nonlinear dynamics of $\mathcal{G}_\ell$ as piecewise affine interval models $\mathcal{H}_\ell$ and the second involves solving the following model discrimination problem for the abstracted models.

Problem 3 (Model Discrimination amongst $\{\mathcal{H}_\ell\}_{\ell=1}^{N_m}$). *Given a sequence of input-output trajectory $\{u_k, y_k\}_{k=0}^{N-1}$, N_m abstracted piecewise affine interval models $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_{N_m}$ and an integer N, determine which model the trajectory belongs to. That is, to find an i that satisfies*

$$\mathcal{B}^N(\mathcal{H}_i) \neq \emptyset \wedge (\mathcal{B}^N(\mathcal{H}_j) = \emptyset, \forall j \in \mathbb{Z}_{N_m}^1, j \neq i). \quad (9)$$

By construction of affine abstraction in Problem 1, we can leverage the fact that $\mathcal{B}^N(\mathcal{G}_\ell) \subseteq \mathcal{B}^N(\mathcal{H}_\ell)$, which indicates that the inconsistent models excluded in Problem 3, i.e., when $\mathcal{B}^N(\mathcal{H}_j) = \emptyset$, are also excluded in Problem 2 because $\mathcal{B}^N(\mathcal{G}_j) \subseteq \mathcal{B}^N(\mathcal{H}_j) = \emptyset$. On the other hand, when $\mathcal{G}_i$ is the true model, then necessarily $\mathcal{B}^N(\mathcal{G}_i) \neq \emptyset$ and also $\mathcal{B}^N(\mathcal{H}_i) \supseteq \mathcal{B}^N(\mathcal{G}_i) \neq \emptyset$. Thus, a solution to Problem 3 also solves Problem 2.

IV. Abstraction and Model Discrimination

In this section, we introduce optimization-based approaches for performing affine abstraction and model discrimination of uncertain nonlinear systems (1). The two methods for solving Problems 1 and 3 (and hence, Problem 2) can be viewed as independent and be used in conjunction with other abstraction or model discrimination approaches.

A. Mesh-Based Abstraction

To solve Problem 1 for the uncertain nonlinear system (1), inspired by the results in [8], [9], we first consider a two-part abstraction approach for a specific subregion I_i. We will subsequently discuss how the multiple subregions are obtained to partition the entire domain of interest to satisfy the desired approximation error. The first part handles the uncertainty in the coefficients A_k, which expands the middle inequality in (7) via enumeration of the vertices of the interval matrix $\mathcal{A}$, as shown in the next lemma.

Each row of the uncertain matrix A_k can be written as a d-dimensional hyperrectangle that is defined as

$$\mathcal{A}_r = [A_{l,r1}, A_{u,r1}] \times \cdots \times [A_{l,rd}, A_{u,rd}], \forall r \in [n]. \quad (10)$$

In the following lemma, the $A_{i,r}$, $B_{i,r}$ and $A_{k,r}$ denote the r-th row of the A_i, B_i and A_k, respectively.

Lemma 1. *Consider the vertex set of the d-dimensional hyperrectangle $\mathcal{A}_r$ represented as $\mathcal{V}_r^A = \{v_{r,1}^A, \ldots v_{r,\rho}^A\}$ with $\rho = |\mathcal{V}^A| \leq 2^d$, where $\rho = 2^d$ holds when $A_{k,r}$ is unstructured, i.e., all elements of $A_{k,r}$ are independent. The constraints*

$$\overline{A}_{i,r} x_k + \overline{B}_{i,r} u_k + \overline{h}_{i,r} \geq (v_{r,q}^A)^T \phi(x_k, u_k), \ \forall q \in [\rho], \quad (11)$$

$$\underline{A}_{i,r} x_k + \underline{B}_{i,r} u_k + \underline{h}_{i,r} \leq (v_{r,q}^A)^T \phi(x_k, u_k), \ \forall q \in [\rho], \quad (12)$$

are equivalent to $\overline{A}_{i,r} x_k + \overline{B}_{i,r} u_k + \overline{h}_{i,r} \geq A_{k,r}\phi(x_k, u_k), \ \forall A_{k,r} \in \mathcal{A}_r$ and $\overline{A}_{i,r} x_k + \overline{B}_{i,r} u_k + \underline{h}_{i,r} \leq A_{k,r}\phi(x_k, u_k), \ \forall A_{k,r} \in \mathcal{A}_r$, for all $(x_k, u_k) \in I_i$.

Proof. This proof follows similar steps to [9, Lemma 1]. Since $\mathcal{A}_r$ is a d-dimensional hyperrectangle with vertex set $\mathcal{V}_r^A = \{v_{r,1}^A, \ldots v_{r,\rho}^A\}$, any point in $A_{k,r} \in \mathcal{A}_r$ can be represented as

$$A_{k,r}^T = \sum_{q=1}^{\rho} \alpha_q v_{r,q}^A, \quad (13)$$

where $\alpha_q \geq 0$ and $\sum_{q=1}^{\rho} \alpha_q = 1$. Multiplying both sides of (11) and (12) by the nonnegative constraint α_q, we have

$$\alpha_q(\overline{A}_{i,r} x_k + \overline{B}_{i,r} u_i + \overline{h}_{i,r}) \geq \alpha_q (v_{r,q}^A)^T \phi(x_k, u_k), \forall q \in [\rho],$$

$$\alpha_q(\underline{A}_{i,r} x_k + \underline{B}_{i,r} u_i + \underline{h}_{i,r}) \leq \alpha_q (v_{r,q}^A)^T \phi(x_k, u_k), \forall q \in [\rho].$$

Adding all of the ρ inequalities above respectively yields

$$\sum_{q=1}^{\rho} \alpha_q(\overline{A}_{i,r} x_k + \overline{B}_{i,r} u_i + \overline{h}_{i,r}) \geq \sum_{q=1}^{\rho} \alpha_q (v_{r,q}^A)^T \phi(x_k, u_k),$$

$$\sum_{q=1}^{\rho} \alpha_q(\underline{A}_{i,r} x_k + \underline{B}_{i,r} u_i + \underline{h}_{i,r}) \leq \sum_{q=1}^{\rho} \alpha_q (v_{r,q}^A)^T \phi(x_k, u_k).$$

In light of $\sum_{q=1}^{\rho} \alpha_q = 1$ and (13), the sufficiency can be obtained directly. Conversely, suppose we have $\overline{A}_{i,r} x_k + \overline{B}_{i,r} u_k + \overline{h}_{i,r} \geq A_{k,r}\phi(x_k, u_k), \ \forall A_{k,r} \in \mathcal{A}_r$ and $\underline{A}_{i,r} x_k + \underline{B}_{i,r} u_k + \underline{h}_{i,r} \leq A_{k,r}\phi(x_k, u_k). \ \forall A_{k,r} \in \mathcal{A}_r$. As the uncertain set $\mathcal{A}_r$ contains every point including all its vertices, thus, (11) and (12) hold. This completes the proof. ☐

The above lemma converts our problem into inequalities for *certain* nonlinear systems (albeit with more inequality constraints). Hence, we can leverage the mesh-based affine abstraction approach in [8] to further recast the affine abstraction problem in Problem 1 into a LP problem.

Theorem 1. *Given a nonlinear function $f : I_i \to \mathbb{R}^n$ with a given* partition $I_i \subset \mathbb{R}^{n+m}$ *for any subregion $I_i \in \mathcal{I}$, let $\mathcal{V} = \{v_1, v_2, \ldots, v_l\}$ be a set of l grid points of a uniform mesh of the subregion I_i and $\mathcal{C} = \{v_1^c, \ldots, v_{2^{(n+m)}}^c\}$ be a set of the corner points of the hyperrectangular domain of I_i. The affine hyperplanes $\overline{f}_i$ and $\underline{f}_i$ that over-approximate/abstract f in domain I_i are given by:*

$$\overline{f}_i = f_{u,i} + \sigma_i, \quad \underline{f}_i = f_{b,i} - \sigma_i,$$

with each r-th element of σ_i defined as $\sigma_{i,r} = \max_{q \in [\rho]} \sigma_{i,r,q}$, where $\sigma_{i,r,q}$ is interpolation error of $(v_{r,q}^A)^T \phi(x_k, u_k)$ according to [8, Propositon 2]. $f_{u,i} = \overline{A}_i x_k + \overline{B}_i u_k + h_{u,i,r}$, and $f_{b,i} = \underline{A}_i x_k + \underline{B}_i u_k + h_{b,i,r}$, where $\overline{A}_i, \underline{A}_i, \overline{B}_i, \underline{B}_i, h_{u,i,r}$ and $h_{b,i,r}$ are obtained from the following linear program-

ming (LP) problem:

$$\begin{aligned} \min_{\theta,\overline{A}_{i,r},\underline{A}_{i,r},\overline{B}_{i,r},\underline{B}_{i,r},h_{u,i,r},h_{b,i,r}} \quad & \theta \\ \text{s.t.} \quad \overline{A}_{i,r}\, x_k + \overline{B}_{i,r}\, u_k + h_{u,i,r} &\geq (v^A_{r,q})^T \phi(x_k, u_k), \quad (14a) \\ \underline{A}_{i,r}\, x_k + \underline{B}_{i,r}\, u_k + h_{b,i,r} &\leq (v^A_{r,q})^T \phi(x_k, u_k), \quad (14b) \\ (\overline{A}_i - \underline{A}_i) x^c_j + (\overline{B}_i - \underline{B}_i) u^c_j + h_{u,i,r} - h_{b,i,r} &\leq \theta \mathbb{1}_n, \quad (14c) \\ \forall k \in [l], \forall j \in [2^{(n+m)}], \forall r \in [n], \forall q \in [\rho]. \end{aligned}$$

Proof. From Lemma 1, the abstraction of original function $f(x_k,\ u_k) = A_k \phi(x_k,\ u_k)$ is equivalent to the abstraction of all $(v^A_{r,q})^T \phi(x_k, u_k)$. Then, following the lines of the proof of Theorem 1 in [8], the above theorem is obtained trivially. □

To reduce the conservativeness, we can partition the domain of interest into multiple subregions and obtain the abstraction by solving the problem in Theorem 1 for each single subregion. The partitioning process can be recursively implemented until the abstraction error in each subregion is smaller than a desired accuracy as shown in Algorithm 1 (see detailed description of this algorithm in [8]).

Algorithm 1: Creating a ε_f-Accurate Partition [8]

Data: f, bound $= \mathcal{X} \times \mathcal{U}$, resolution r, desired accuracy ε_f

function epsPartition(f, bound, r, ε_f)
- $(\overline{f}, \underline{f}, e(\overline{f}, \underline{f})) \leftarrow$ abstraction(f, bound, r, ε_f)
- **if** $e(\overline{f}, \underline{f}) \leq \varepsilon_f$ **then**
  - partition $= \{\overline{f}, \underline{f}, \text{bound}\}$
  - **return** (partition)
- **else**
  - $\mathcal{I} \leftarrow$ divBounds(bound)
  - **for** $i = 1 : 2^{n+m}$ **do**
    - cell$\{i\}$ = epsPartition(f, I_i, r, ε_f)
  - **end**
  - partition $= \bigoplus_{i=1}^{2^{n+m}} \{\text{cell}\{i\}\}$ ($\bigoplus$ = concatenation)
- **end**
- **return** (partition, $\mathcal{I}$)

function divBounds(bound)
- Refer to Section IV-A for its description
- **return** (subBounds)

function abstraction(f, bound, r, ε_f)
- Refer to *Theorem 1* for its description
- **return** $(\overline{f}, \underline{f}, e(\overline{f}, \underline{f}))$

B. Model Discrimination

In Problem 2, we will assume the following:

Assumption 1. *The length-N input-output trajectories are only consistent with one uncertain nonlinear model. Thus, we must have $\mathcal{B}^N(\mathcal{G}_i) \cap \mathcal{B}^N(\mathcal{G}_j) = \emptyset$ for all $i \neq j$.*

Assumption 2. *The subregion I_i is a closed bounded region for $(x_k, u_k) \in I_i$, and its bounds can be described as following constraints with c_i constraints:*

$$S_i x_k + T_i u_k \leq \beta_i. \tag{15}$$

where S_i, T_i and β_i are real matrices/vectors.

Our optimization-based model discrimination approach is based on model invalidation that eliminates all models that are incompatible with the observed length-N input-output trajectory. Since we assume that only one original uncertain nonlinear model can be consistent, with a sufficiently accurate affine abstraction, i.e., with small enough ε_f, we can also assume that the length-N input-output trajectory is only compatible with one abstracted model. Using this fact, we propose the following model invalidation algorithm that (in)validates a specific piecewise affine interval model $\mathcal{H}_\ell$:

Theorem 2. *Given an abstracted piecewise affine interval model $\mathcal{H}_\ell$ and a length-N input-output sequence $\{u_k, y_k\}_{k=0}^{k=N-1}$, the model is invalidated if the following problem is infeasible:*

Find $x_k, \omega_k, a_{i,k}, s_{i,k},\ \forall k \in \mathbb{Z}^0_{N-1}, \forall i \in [p]$
subject to $\forall k \in \mathbb{Z}^0_{N-1}, \forall i \in [p]$:

$$\begin{aligned} & x_{k+1} \leq \overline{A}_i x_k + \overline{B}_i u_k + \overline{h}_i + \omega_k + s_{i,k} \mathbb{1}_n, && (16a) \\ & x_{k+1} \geq \underline{A}_i x_k + \underline{B}_i u_k + \underline{h}_i + \omega_k + s_{i,k} \mathbb{1}_n, && (16b) \\ & S_i x_k + T_i u_k \leq \beta_i + s_{i,k} \mathbb{1}_{c_i}, && (16c) \\ & y_k = C_k x_k + \eta_k, && (16d) \\ & a_{i,k} \in \{0, 1\},\ \textstyle\sum_{i \in [p]} a_{i,k} = 1, && (16e) \\ & \|\omega_k\| \leq \varepsilon_\omega,\ \|v_k\| \leq \varepsilon_v,\ (a_{i,k}, s_{i,k}) : SOS\text{-}1, && (16f) \end{aligned}$$

where $s_{i,k}$ is a slack variable that is free when $a_{i,k}$ is zero and zero otherwise (by virtue of the special ordered set of degree 1 (SOS-1) constraint).

Proof. $a_{i,k} = 1$ implies that $S_i x_k + T_i u_k \leq \beta_i$ holds and its corresponding constraints (16a)–(16c) hold since the SOS-1 constraint ensures that $s_{i,k} = 0$, which means that the state x_{k+1} must be bounded by the given abstraction model if $(x_k,\ u_k) \in I_i$. On the contrary, the $s_{i,k}$ is free if $a_{i,k} = 0$ and (16a)–(16c) hold trivially. Moreover, due to the constraint in (16e), only one $a_{i,k} = 1$ for all $i \in [p]$ is possible, which means that only one partition is valid. Finally, if the above optimization problem is infeasible, it means that the input-output sequence $\{u_k, y_k\}_{k=0}^{k=N-1}$ cannot be consistent with the length-N behavior of $\mathcal{H}_\ell$, i.e., $\{u_k, y_k\}_{k=0}^{k=N-1} \notin \mathcal{B}^N(\mathcal{H}_\ell)$, hence the model is invalidated. □

Next, to solve the model discrimination problem, we can leverage the model invalidation approach above to eliminate all inconsistent models. Since only one model can be compatible by Assumption 1, model discrimination can be achieved when all other inconsistent models are eliminated except for the true model. This model discrimination process is summarized in Algorithm 2. As the time horizon k is increased to N, it is guaranteed to discriminate against all false models by Assumption 1. The determination of N that can guarantee model discrimination is called T-detectability

Algorithm 2: Model Discrimination with Length k

Data: Models $\mathcal{G}_1 \ \ldots \mathcal{G}_{N_f}$, Input-Output Sequence $= \{u_\ell, y_\ell\}_{\ell=0}^{\ell=k-1}$

function findModel($\mathcal{G}_1 \ \ldots \mathcal{G}_{N_f}$, $\{u_\ell, y_\ell\}_{\ell=0}^{\ell=k-1}$)
- valid $\leftarrow [N_f]$
- **for** $i = 1 : N_f$ **do**
  - Check Feasibility of Theorem 2
  - **if** *infeasible* **then**
    - Remove i from valid
  - **end**
- **end**
- **return** valid

in the literature [11] and will be the subject of future work.

V. Simulation Results

In this section, we demonstrate the proposed approaches for affine abstraction and model discrimination for swarm intent/formation estimation. All simulations are implemented in MATLAB on a 2.2 GHz machine with 16 GB of memory.

A. Dynamic Models

The dynamics of each swarm agent is described by the Dubins Car model [19]:

$$p_{x,k+1} = p_{x,k} + u_s \cos(\theta_k)\delta t + w_{px,k}, \tag{17a}$$

$$p_{y,k+1} = p_{y,k} + u_s \sin(\theta_k)\delta t + w_{py,k}, \tag{17b}$$

$$\theta_{k+1} = \theta_k + \frac{u_s}{L}\tan(u_\phi)\delta t + w_{\theta,k}, \tag{17c}$$

where the p_x and p_y represents the position of the agent and θ is the heading angle of the agent, all of which are considered as system states, L is the length between the front and rear tires and is set to $1.5m$, u_s is the speed of the agent and is assumed to be in the range of $[0.95, 1.05]\frac{m}{s}$, which introduces parametric uncertainties, sampling time δt is set to $0.1s$, $w_{px,k}$, $w_{py,k}$ and $w_{\theta,k}$ represent process noise or heterogeneity among the agents and are set to be within $|w_{px,k}| \leq 0.01$, $|w_{py,k}| \leq 0.01$ and $|w_{\theta,k}| \leq 0.0067$, respectively. Further, we assume that we observe all system states with measurement noise signals setting to be $|v_{p_x}| \leq 0.01$, $|v_{p_y}| \leq 0.01$ and $|v_\theta| \leq 0.004$, respectively. In addition, a reference signal $\theta_{desired}$ based on the centroid of the swarm formation (c_x, c_y) is assumed to be given:

$$\theta_{desired} = \arctan 2(c_y - p_y, c_x - p_x), \tag{18}$$

which the agents utilize for feedback control according to the following proportional control law:

$$u_\phi = \min(\frac{\pi}{8}, \max(-\frac{\pi}{8}, K_p(\theta_{desired} - \theta))), \tag{19}$$

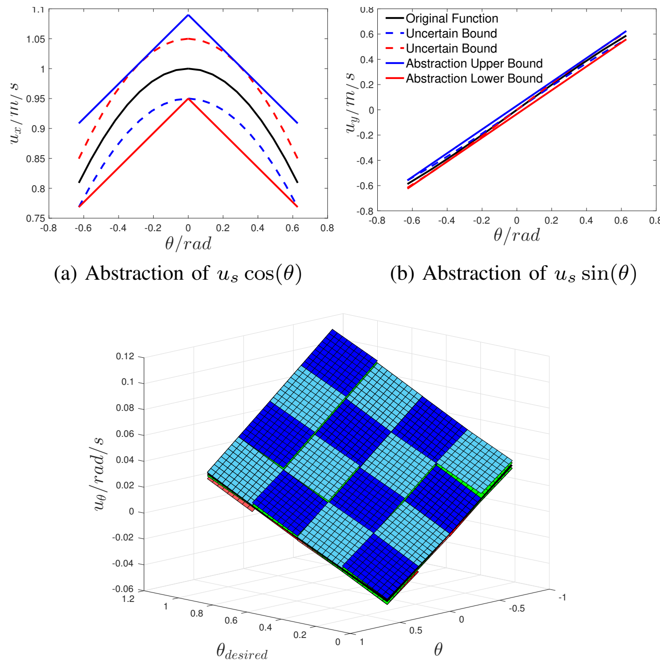
where the saturation functions ensure that the steering angle of each agent never exceeds $[-\frac{\pi}{8}, \frac{\pi}{8}]$ rad.

We consider three swarm intents or formations, which are dependent on the choice of the K_p value. When $K_p = 0.1$ (Model I), the swarm intends to move towards the centroid of the swarm, while when $K_p = -0.1$ (Model II), the swarm moves away from the centroid. Further, we also consider a third intent with $K_p = 0$ (Model III) where the swarm agents do not interact with each other.

For implementation of both abstraction and model discrimination, we used Yalmip [20] and Gurobi [21].

B. Affine Abstraction Results

First, we apply our affine abstraction algorithm to the system dynamics (17), specifically, the uncertain nonlinear parts of the dynamics involving $u_s\cos(\theta)$, $u_s\sin(\theta)$ and $\frac{u_s}{L}\tan(u_\phi)$, where u_ϕ is given by (19). The former two functions are defined on the domain of $\theta \in [-\frac{\pi}{5}, \frac{\pi}{5}]$, while the third is defined in the domain of $\theta \times \theta_{desired} \in [-\frac{\pi}{5}, \frac{\pi}{5}] \times [0, \frac{\pi}{3}]$. Further, since these functions are Lipschitz continuous on the given interval, the interpolation error of the abstraction approach is well defined according to Theorem 1. The desired accuracy is set to be $\varepsilon_{f,x} = 0.3$ for the $u_s\cos(\theta)$,



(a) Abstraction of $u_s\cos(\theta)$ (b) Abstraction of $u_s\sin(\theta)$

(c) Abstraction of $\frac{u_s}{L}\tan(u_\phi(\theta, \theta_{desired}))$

Fig. 1: Illustration of affine abstraction of the uncertain nonlinear functions $u_s\cos(\theta)$, $u_s\sin(\theta)$ and $\frac{u_s}{L}\tan(u_\phi)$.

$\varepsilon_{f,y} = 0.3$ for $u_s\sin(\theta)$ and $\varepsilon_{f,\theta} = 0.02$ for $\frac{u_s}{L}\tan(u_\phi)$. The result of the affine abstraction is shown in Figure 1.

From Figure 1, we observe that the resulting affine hyperplanes envelop the uncertain nonlinear dynamics on the defined domains of interest and the minimum approximation error is obtained, as desired.

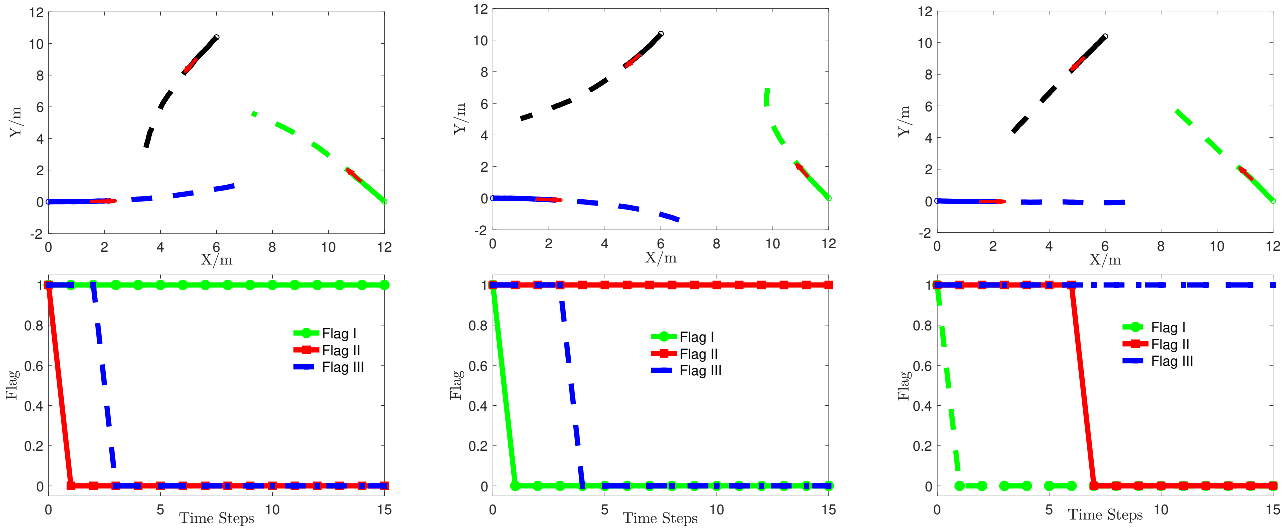
C. Model Discrimination Results

Next, the model discrimination algorithm is applied to the abstracted models from Section V-B using sampled input-output trajectories of 15 time steps, where the input in this case is $\theta_{desired}$ and the outputs are p_x, p_y, θ. These outputs can all be generated using the system dynamics from (17) with the following initial conditions for 3 agents: $[0, 0, 0]$, $[12, 0, \frac{\pi}{3}]$ and $[6, 4\sqrt{3}, -\frac{2\pi}{3}]$ representing $p_{x,0}$, $p_{y,0}$ and θ_0, respectively. Figure 2 illustrates how the true model of the system can be detected and we observe that the true model can be discriminated within 10 time steps (i.e., 1 second).

Next, we vary $\varepsilon_{f,x}$, $\varepsilon_{f,y}$ and $\varepsilon_{f,\theta}$ to investigate their effects on the maximum number of time steps needed for model discrimination. The results are tabulated in Table I. We observe that Model III is relatively hard to be discriminated from the other models and that while a smaller ε_f provides better abstraction and thus, makes model discrimination easier, it is also interesting to note that no tangible advantage is gained by further decreasing ε_f beyond a certain threshold.

TABLE I: Required Number of Time Steps for Model Discrimination as a Function of Accuracy ε_f.

	Model I	Model II	Model III
$\varepsilon_{f,x} = 0.3$, $\varepsilon_{f,y} = 0.3$, $\varepsilon_{f,\theta} = 0.5$	4	4	10
$\varepsilon_{f,x} = 0.3$, $\varepsilon_{f,y} = 0.3$, $\varepsilon_{f,\theta} = 0.02$	4	4	7
$\varepsilon_{f,x} = 0.3$, $\varepsilon_{f,y} = 0.3$, $\varepsilon_{f,\theta} = 0.015$	4	4	7



(a) Model I ($K_p = 0.1$) (b) Model II ($K_p = -0.1$) (c) Model III ($K_p = 0$)

Fig. 2: Sampled state trajectories (top row) and the corresponding model discrimination results (bottom row). In the state trajectory figure, the solid lines denote the trajectories of three models within the first 15 steps, red arrows denote the current heading angles at the 15-th steps and the dash lines denote the future trajectories after 15 steps. In the (bottom) figures depicting the model discrimination results, for $i \in \{\text{I, II, III}\}$, Flag i is 1 when the corresponding model i is validated and is 0, if invalidated. Model discrimination is achieved when only one Flag is 1.

VI. Conclusion

We proposed optimization-based approaches for affine abstraction and model discrimination of uncertain nonlinear systems, where the uncertain nonlinear system of interest is a linear combination of nonlinear basis functions and bounded uncertain parameters/coefficients. First, a mesh-based affine abstraction method is introduced to over-approximate the complex nonlinear dynamics by two affine hyperplanes that bracket all original system behaviors in each subregion. Then, we proposed a model discrimination approach based on model invalidation for piecewise affine interval models that are obtained from the abstraction method, which can be solved as an MILP. Finally, we demonstrated our approaches on an example of intent/formation identification of autonomous swarm systems.

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