Time-Varying Tube-Based Output Feedback MPC for Constrained Linear Systems with Intermittently Delayed Data [⋆]

Syed M. Hassaan [*] **Tarun Pati** [*] **Qiang Shen** [**] **Sze Zheng Yong** [*]

[*] *School for Engineering of Matter, Transport and Energy, Arizona State University, Tempe, AZ, USA (e-mail: {shassaan,tpati,szyong}@asu.edu).*
[**] *School of Aeronautics and Astronautics, Shanghai Jiao Tong University, Shanghai, P.R. China (e-mail: qiangshen@sjtu.edu.cn)*

Abstract: This paper proposes a time-varying tube-based output feedback model predictive control (MPC) design for constrained linear systems in the presence of intermittently delayed observations, where the delayed/missing data patterns for each period satisfy a finite-length language. The design consists of a dynamic state estimator whose estimation errors satisfy equalized recovery (a weaker form of invariance with time-varying finite bounds), as well as an output feedback control law that extends existing tube-based output feedback MPC approaches to allow time-varying tubes for tightening the original state and input constraints. The resulting time-varying tube-based output feedback MPC design is robust to time-varying disturbances and errors, including when the observations are intermittently delayed. Further, we provide sufficient conditions for recursive feasibility and robust exponential stability of the proposed design. Simulation results demonstrate that the proposed approach is able to robustly stabilize and control a constrained linear system despite disturbances, noise and missing/delayed data.

Keywords: Output-based control; Robust model predictive control; Missing and delayed data; Robust time-varying tubes; Optimal control and optimization

1. INTRODUCTION

Model predictive control (MPC) is a model-based control strategy that solves an optimal control problem *on-line*, which has lately garnered extensive attention due to its ability to handle nonlinearity and explicitly account for constraints. However, earlier works on MPC often required noiseless full state information and fully known system dynamics, e.g., Findeisen et al. (2003). Often, only noisy measurements of a subset of the states are available, while the system dynamics is affected by disturbances/process noise. Certainty equivalence and separation principle can unfortunately not be used to guarantee closed-loop stability due to nonlinearity of the controller for the constrained MPC. Furthermore, sensor data may not always be available, especially in networked systems where packet delays and drops may be inevitable, thus MPC may not be inherently robust (Sakthivel et al. (2018)). This motivates us to develop a tube-based output feedback MPC strategy that is robust to missing and delayed data patterns, is recursively feasible and makes the system robustly exponentially stable to some bounded set.

Literature Review. Survey papers (e.g., Borrelli et al. (2017) and references therein) categorized robust model predictive control into (i) open-loop min-max approaches, where the optimal control sequence is computed based on open-loop predictions under the worst-case uncertainty, e.g., Bemporad et al. (2003), (ii) feedback approaches, where the robust control policy is optimized, e.g., Kerrigan and Maciejowski (2004), and (iii) tube-based model predictive controllers, which combine open-loop prediction based on the nominal system and a disturbance/error invariance feedback control, e.g., Raković et al. (2012). Open-loop approaches may be very conservative and feedback approaches are often computationally intractable except for some special cases, while tube-based methods provide a trade-off between optimality and computation by moving the more intensive computation offline. Earlier tube-based approaches assume full state information, and more recent works on output-based methods consider the combination of state estimation with a robust MPC framework, e.g., Copp and Hespanha (2014). However, these approaches do not directly apply when there is missing or delayed data, as is the case we are considering.

Thus, another relevant literature pertains to controller and estimator designs for systems with missing/delayed data, which have been studied for networked control systems (e.g., Sinopoli et al. (2004)) and for security problems taking false data injection attacks into account (e.g., Fawzi et al. (2014); Yong et al. (2018)), mainly in a stochastic setting. More recently, robust controllers and estimators with worst-case bounds have been proposed for the missing/delayed data scenario in Rutledge et al. (2018, 2020); Hassaan et al. (2021).

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Contributions. In this paper, we propose a time-varying tube-based output feedback MPC for constrained linear systems in the presence of delayed and missing observations, whose patterns satisfy a periodic finite-length language. Our design is based on an equalized recovery framework (Rutledge et al. (2018, 2020); Hassaan et al. (2021)), which is a relaxation of robust (controlled) invariance and related to N-step recurrence (Fiacchini and Alamir (2018)). The basic idea proposed is to use a dynamic state estimator that achieves equalized recovery (thus, guaranteeing finite time-varying estimation error bounds), which is then used to find an appropriate control law using an extension of tube-based output feedback MPC designs that allows time-varying tubes for tightening the state and input constraints. Specifically, the contributions are:

(1) A novel time-varying tube-based output feedback MPC design that is robust to time-varying disturbances and errors, including the case of intermittently delayed observations whose patterns satisfy a periodic finite-length language specification;
(2) A formal analysis of recursive feasibility and stability properties for the proposed robust MPC design.

The simulation results demonstrate that the proposed approach is able to robustly stabilize or control a constrained linear system despite missing or delayed data.

2. PROBLEM FORMULATION

2.1 Notation and Definitions

Notation: $\mathbb{R}^n$ represents the n-dimensional Euclidean space, $\mathbb{Z}$ denotes the set of non-negative integers and $\mathbb{Z}_a^b$ denotes the set of integers from a through b. The operators $\div$, $\oplus$ and $\ominus$ denote the modulo operation, Minkowski sum of sets and Pontryagin difference of sets, respectively, while $\|\cdot\|$ is used to denote the infinity norm of vectors. An identity matrix of size s is denoted by I_s, a vector of ones of length s is denoted by $\mathbb{1}_s$, while a zero matrix of dimension a-by-b is denoted by $0_{a\times b}$. The inequalities for comparing vectors and matrices are all element-wise.

In contrast to conventional tube-based approaches where invariant sets are used as "fixed" tubes, we consider time-varying tubes based on a weaker requirement called *equalized recovery* (Rutledge et al. (2020); Hassaan et al. (2021)), defined below, that is often easier to compute and has been shown to be effective in the presence of missing and delayed data. This concept is also closely related to N-step recurrent sets in Fiacchini and Alamir (2018).

Definition 1. (Equalized Recovery). A set $S_0 \triangleq \{x \in \mathbb{R}^n \mid \|x\| \leq \mu_0\}$ is said to satisfy an equalized recovery level μ_0 with recovery time T_d and intermediate levels $\mu_i \geq \mu_0$, $\forall i \in \mathbb{Z}_1^{T_d-1}$ for the discrete-time control system $x_{k+1} = A_k x_k + B_k u_k + W_k w_k$, if for all initial states $x_0 \in S_0$, there exists a sequence of control inputs $\{u_i\}_{i=0}^{T_d-1} \in \mathcal{U}$ such that the future state at $k = T_d$ satisfies $x_{T_d} \in S_0$ for all $w_k \in \mathcal{W}$ and all intermediate states satisfy $x_i \in S_i \triangleq \{x \in \mathbb{R}^n \mid \|x\| \leq \mu_i\}$ for all $i \in \mathbb{Z}_1^{T_d-1}$.

Further, we define the notion of robust exponential stability that we will prove later for our proposed design.

Definition 2. (Robust Exponential Stability). A set S is robustly exponentially stable (Lyapunov stable and exponentially attractive) for the system $x_{k+1} = A_k x_k + B_k u_k + W_k w_k$, $w_k \in \mathcal{W}$, where $\mathcal{W}$ is bounded, with a region of attraction $\mathbb{X}$ if there exist $c > 0$ and $\delta \in (0,1)$ such that any solution of $\{x_i\}_{i\in\mathbb{Z}}$ with initial state $x_0 \in \mathbb{X}$, and any admissible disturbance sequence $\{w_i\}$ satisfies the Hausdorff distance $d(x_i, S) \leq c\delta^i d(x_0, S)$ for all $i \in \mathbb{Z}$.

2.2 System Dynamics and Delayed Data Model

System Dynamics: We consider a constrained discrete-time linear time-varying system that is subjected to bounded process and measurement noise given by:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k + W_k w_k, \\ z_k &= C_k x_k + V_k v_k, \\ Y_k &= \{z_{k-\omega(i)} | i + \omega(i) = k, i \leq k\}, \end{aligned} \tag{1}$$

where $x_k \in \mathcal{X} \subseteq \mathbb{R}^n$ is the system state at time k, $u_k \in \mathcal{U} \subseteq \mathbb{R}^m$ is the input to the system, $w_k \in \mathcal{W} \subseteq \mathbb{R}^{n_w}$ is the process noise, $v_k \in \mathcal{V} \subseteq \mathbb{R}^{n_v}$ is the measurement noise, $z_k \in \mathbb{R}^p$ is the time-stamped observation or measurement that is possibly affected by a time-delay attack or any naturally occurring delay, $Y_k \subset \mathbb{R}^p$ is the set of all measurement data that is received at time step k, $\omega(i)$ is the time delay associated with z_i at the time step i and satisfies $\omega(i) \in \mathbb{Z}_0^{\overline{\omega}}$, and $\overline{\omega}$ is an upper bound on the number of time steps that a packet is delayed by. The discrete variable $\omega(i) = 0$ denotes that the i-th measurement is received without delay, while $\omega(i) = a$ implies that the i-th measurement is delayed by a steps. The system matrices A_k, B_k, C_k, W_k, V_k and f_k are all known and of appropriate dimensions. We assume that the process and measurement noises w_k and v_k are unknown but bounded with $w_k \in \mathcal{W} = \{w \in \mathbb{R}^{n_w} | \|w\| \leq \eta_w\}$ and $v_k \in \mathcal{V} = \{v \in \mathbb{R}^{n_v} | \|v\| \leq \eta_v\}$ for every time step k, while the control input u_k is bounded with $u_k \in \mathcal{U} = \{u \in \mathbb{R}^m \mid \|u\| \leq \eta_u\}$ and the system state x_k is constrained with $x_k \in \mathcal{X} = \{x \in \mathbb{R}^n | \|x\| \leq \eta_x\}$. Without loss of generality, we assume that the initial time is $k = 0$.

Delayed Data Model: Intermittently delayed/missing data patterns in each time period T_d (assumed to be known) are restricted to a set expressed by fixed-length language specifications, e.g., 'the i-th observation is delayed by at most m time steps' or 'at most m delayed/missing measurements in a fixed interval.' The delayed data patterns can be the result of naturally occurring delays or packet dropouts due to communication network congestion or losses, or caused by deliberate (cyber) attacks by an adversary or hacker. Formally, our delayed data model is a fixed-length language $\mathcal{L} = \{\mathcal{W}_\alpha\}_{\alpha=1}^{|\mathcal{L}|}$ with period T_d that specifies the set of allowable delay mode sequences $\omega_\alpha(qT_d)\omega_\alpha(1 + qT_d)\omega_\alpha(2 + qT_d)\ldots\omega_\alpha((q + 1)T_d - 1)$ for all $q \in \mathbb{N}$, where the α-th possible sequence is called a word $\mathcal{W}_\alpha$ Note that $i + \omega(i) > T_d$ means that the i-th measurement is delayed beyond the horizon T_d, which is assumed to be equivalent to the situation that the i-th measurement is missing. In other words, missing data can be considered as a special case of delayed data in our setting.

To use the information from the delayed-data model, we use the *reduced event-based language* framework from Hassaan et al. (2021). For conciseness, we now directly introduce the reduced event-based language. Interested readers can find the detailed process of obtaining the reduced event-based language from a given delayed-data model in Hassaan et al. (2021).

Definition 3. (Reduced Event-Based Language). Given a delayed data model as a fixed-length language $\mathcal{L}$ of period T_d, a reduced event-based language $\mathcal{L}^{E'} = \{\mathcal{E}'_\alpha\}_{\alpha=1}^{|\mathcal{L}^{E'}|}$ is a set that translates the words $\mathcal{W}_\alpha \in \mathcal{L}$ into a series of binary sequences $\mathcal{E}'_\alpha$, called an *event sequence*, that captures the history of available data at each step for each word $\mathcal{W}_\alpha$.

2.3 Problem Statement

Problem 1. Given the system dynamics (1), a delayed data model specified by a language $\mathcal{L}$ of period T_d, design a time-varying tube-based output feedback model predictive controller (MPC) that is recursively feasible and can robustly stabilize the system (1).

3. MAIN RESULTS

3.1 Tube-Based Output MPC Approach

The objective of this paper is to design a tube-based output feedback MPC for robustly controlling the system (1) in the presence of missing and delayed data that satisfy a language $\mathcal{L}$ of period T_d, i.e., to solve Problem 1. Similar to other tube-based designs when there is no missing or delayed data, our proposed solution consists of three parts:

(i) A *dynamic state estimator* inspired by Hassaan et al. (2021) of the form:

$$\begin{aligned} \hat{x}_{k+1} &= A_k \hat{x}_k + B_k u_k - u^e_k, \\ s_{k+1} &= A_k s_k + u^e_k + L^{\mathcal{E}'_\alpha}_{k \div T_d} \tilde{z}_k, \end{aligned} \quad (2)$$

with $\tilde{z}_k = \begin{cases} z_k - C_k(\hat{x}_k + s_k), & \text{if } z_k \in \bigcup_{j=0}^{k} Y_j, \\ 0, & \text{otherwise,} \end{cases}$

where $\hat{x}_k$ is the estimate of x_k, s_k is an auxiliary state estimating the estimation error $\tilde{x}_k = x_k - \hat{x}_k$ and $L^{\mathcal{E}'_\alpha}_{k \div T_d} \in \mathbb{R}^{n \times p}$ is the Luenberger gain at the $(k \div T_d)$-th step as a function of the observed prefix of $\mathcal{E}'_\alpha$ within the same fixed-length horizon T_d. The vector $u^e_k \in \mathbb{R}^n$ is the *causal* estimator output error injection term given by:

$$u^e_k = \nu^{\mathcal{E}'_\alpha}_{k \div T_d} + \textstyle\sum_{i=0}^{k \div T_d} M^{\mathcal{E}'_\alpha}_{(k \div T_d, i)} \tilde{z}_i, \quad (3)$$

where $M^{\mathcal{E}'_\alpha}_{(k \div T_d, i)} \in \mathbb{R}^{n \times p}$ and $\nu^{\mathcal{E}'_\alpha}_{k \div T_d} \in \mathbb{R}^n$ are gain matrices at the $(k \div T_d)$-th step within a fixed-length horizon T_d, which are also a function of the observed prefix of $\mathcal{E}'_\alpha$ within the same fixed-length horizon T_d. The gain matrices $(M^{\mathcal{E}'_\alpha}_{(k \div T_d, i)}, L^{\mathcal{E}'_\alpha}_{k \div T_d}, \nu^{\mathcal{E}'_\alpha}_{k \div T_d})$ will be designed in Section 3.1.1 to satisfy equalized recovery despite missing and delayed data. The gains are to be designed offline, and implemented in the resulting estimator (2)–(3) at run-time.

(ii) A *tracking controller* to minimize the difference between the actual states of the system (1) and their nominal states $\bar{x}_k$ corresponding to the system (1):

$$\bar{x}_{k+1} = A_k \bar{x}_k + B_k \bar{u}_k, \quad (4)$$

where $\bar{x}_k \in \mathbb{R}^n$ and $\bar{u}_k \in \mathbb{R}^m$ are the nominal state and nominal input, respectively. Specifically, we consider a causal affine feedback tracking controller that also satisfies equalized recovery, of the form:

$$u^c_k = \textstyle\sum_{i=0}^{k \div T_d} K_{(k \div T_d, i)}(\hat{x}_i - \bar{x}_i) + \lambda_{k \div T_d}, \quad (5)$$

where $K_{(k \div T_d, i)}$ and $\lambda_{k \div T_d}$ are feedback gains that will be designed in Section 3.1.2. Similar to the estimator in part (i), the gains in (5) are to be designed offline and u^c_k is implemented at run-time.

(iii) A *nominal MPC with time-varying tubes* and a prediction horizon T_p for the nominal system (4), which is subjected to tighter control and state constraints as well as a terminal constraint:

$$\bar{x}_k \in \mathcal{X} \ominus S^c_{k \div T_d}, \ \bar{x}_{k+T_p} \in \mathcal{X}_f, \quad (6)$$

$$\bar{u}_k \in \mathcal{U} \ominus (\{\lambda_k\} \oplus \bigoplus_{i=0}^{k \div T_d} K_{(k \div T_d, i)}(S^c_i \oplus (-S^e_i))), \quad (7)$$

where S^e_k and S^c_k are time-varying bounds/tubes for the estimation and tracking errors that are obtained in Sections 3.1.1 and 3.1.2, respectively, while $\mathcal{X}_f$ will be defined in Section 3.2. Solving the nominal MPC problem in Section 3.1.3 at each time step k will generate optimal sequences of nominal control inputs $\bar{\mathbf{u}}^*_k = \{\bar{u}^{k*}_k, \bar{u}^{k*}_{k+1}, \bar{u}^{k*}_{k+2}, ..., \bar{u}^{k*}_{k+T_p-1}\}$ and states $\bar{\mathbf{x}}^*_k = \{\bar{x}^{k*}_k, \bar{x}^{k*}_{k+1}, \bar{x}^{k*}_{k+2}, ..., \bar{x}^{k*}_{k+T_p}\}$, and the first nominal input $\bar{u}^{0*}_k$ and state $\bar{x}^{0*}_k$ are taken as the current nominal input $\bar{u}_k$ and state $\bar{x}_k$, respectively, i.e.,

$$\bar{u}_k = \bar{u}^{k*}_k, \quad \bar{x}_k = \bar{x}^{k*}_k. \quad (8)$$

By combining the three components above, the resulting (run-time) feedback law for the proposed tube-based output feedback MPC with missing/delayed data is given by

$$u_k = \bar{u}_k + u^c_k, \quad (9)$$

with $\bar{u}_k$ and u^c_k given in (8) and (5), respectively.

Next, we describe in detail the design of the estimator and controller gains, as well as the nominal MPC problem.

3.1.1. Dynamic State Estimator: Using the dynamic state estimator in (2), the estimation error system with states $\tilde{x}_k \triangleq x_k - \hat{x}_k$ and s_k can be written as:

$$\begin{bmatrix} \tilde{x}_{k+1} \\ s_{k+1} \end{bmatrix} = \begin{bmatrix} A_k & 0_{n \times n} \\ 0_{n \times n} & A_k \end{bmatrix} \begin{bmatrix} \tilde{x}_k \\ s_k \end{bmatrix} + \begin{bmatrix} I_n \\ I_n \end{bmatrix} u^e_k + \begin{bmatrix} W_k w_k \\ L^{\mathcal{E}'_\alpha}_{k \div T_d} \tilde{z}_k \end{bmatrix}, \quad (10)$$

$$\tilde{z}_k = \begin{cases} \begin{bmatrix} C_k & -C_k \end{bmatrix} \begin{bmatrix} \tilde{x}_k \\ s_k \end{bmatrix} + V_k v_k, & \text{if } z_k \in \bigcup_{j=0}^{k} Y_j, \\ 0, & \text{otherwise.} \end{cases} \quad (11)$$

Then, the problem of finding estimator gains in (2)–(3) is the same as proposed in Hassaan et al. (2021), the solution of which will yield minimized estimation error guarantees $\{\mu^e_i\}_{i=0}^{T_d-1}$ as well as path-dependent gains M^α, ν^α, and L^α as stacked forms of the gains in (2)–(3). The structure of these stacked gains is given in Appendix A. Further, using the designed dynamic state estimator and considering that we can use the dynamic finite-horizon estimator in a periodic fashion owing to the periodic nature of the missing/delayed data model, we can guarantee that the state estimation error satisfies equalized recovery (cf. Definition 1), i.e., the state estimation errors $\tilde{x}_k = x_k - \hat{x}_k$ for all $k \in \mathbb{Z}$ satisfy time-varying estimator tubes:

$$\tilde{x}_k \in S^e_{k \div T_d} \triangleq \{x \mid \|x\| \leq \mu^e_{k \div T_d}\}, \quad (12)$$

where we consider the worst-case estimation error bound $\mu^e_{k \div T_d} \triangleq \max_{\alpha \in \mathbb{Z}_1^{|\mathcal{L}^{E'}|}} \mu^{e,\alpha}_{k \div T_d}$ at each step $(k \div T_d)$ in the nominal MPC problem in Section 3.1.3 since future delay patterns are not available in a predictive horizon.

3.1.2. Tracking Controller: The goal of the tracking controller is to find a (bounded) time-varying tube around the nominal state trajectory obtained by the nominal MPC in Section 3.1.3 that, similar to the dynamic state estimator, also satisfies equalized recovery (cf. Definition 1). This controller can also be viewed as a disturbance rejection controller since it must robustly satisfy equalized recovery despite the presence of "disturbances" in the form of process noise w_k and state estimation error $\tilde{x}_k$.

To design this controller, we first find the tracking error system corresponding to the system (1) and its nominal form (4), along with the control input of the form (9). Specifically, it can be shown that the difference between the actual state and the nominal state, i.e., $e_k \triangleq x_k - \bar{x}_k$ leads to a tracking error system with state e_k given below:

$$e_{k+1} = A_k e_k + B_k u_k^c + W_k w_k,\ y_k^c = e_k - \tilde{x}_k, \qquad (13)$$

where $\tilde{x}_k$ is considered a "noise" term with known time-varying bound that is obtained from (12), while the control input u_k^c is the affine feedback law in (5), which can also be expressed in terms of the state e_k and "noise" $\tilde{x}_k$ as:

$$\begin{aligned} u_k^c &\triangleq \textstyle\sum_{i=0}^{k\div T_d} K_{(k\div T_d,i)}(\hat{x}_i - \bar{x}_i) + \lambda_{k\div T_d} \\ &= \textstyle\sum_{i=0}^{k\div T_d} K_{(k\div T_d,i)}(e_i - \tilde{x}_i) + \lambda_{k\div T_d}, \end{aligned}$$

and the output is chosen/computed as the difference between the state estimate $\hat{x}_k$ in (2) and the nominal state $\bar{x}_k$ in (8), given by $y_k^c \triangleq \hat{x}_k - \bar{x}_k = e_k - \tilde{x}_k$, which is "available" at each time step, regardless of whether the actual outputs of the system (1) are available or not. In this case, both approaches in Rutledge et al. (2020); Hassaan et al. (2021) are equivalent and applicable. Here, we utilize the approach in Rutledge et al. (2020) that requires less decision variables to find the controller gains $K_{(k\div T_d,i)}$ and $\lambda_{k\div T_d}$ using Q-parameterization.

Lemma 4. (Tracking Controller Design). For the system (1) and its nominal counterpart (4) as well as a dynamic state estimator (2), a finite-horizon tracking controller for each period given in (5) can achieve equalized recovery with minimized recovery levels $\{\mu_i^c\}_{i=0}^{T_d-1}$ for the tracking errors corresponding to (13) if the following is feasible:

$$\begin{aligned} &\min_{Q,r,\mu^c} J(\mu^c) \\ &\text{s.t. } Q \text{ is } m\text{-by-}n \text{ block lower triangular,} \\ &\quad \forall(\|w\| \le \eta_w, \|\tilde{x}_i\| \le \mu_i^e, \forall i \in \mathbb{Z}_0^{T_d-1}, \|e_0\| \le \mu_0^c): \\ &\quad \|e_{T_d}\| \le \mu_0^c, \mu_0^c \ge 0, \|e_i\| \le \mu_i^c, \mu_i^c \ge \mu_0^c, \forall i \in \mathbb{Z}_0^{T_d-1}, \\ &\quad e = \Theta^c w - HBQ\tilde{x} + \Xi^c e_0 + HBr, \end{aligned} \qquad (14)$$

where

$$\begin{aligned} \mu^e &= \left[\mu_0^e, \mu_1^e, \ldots, \mu_{T_d-1}^e\right]^\top, \mu^c = \left[\mu_0^c, \mu_1^c, \ldots, \mu_{T_d-1}^c\right]^\top, \\ \bar{I} &= \left[I_{nT_d}\ 0_{nT_d\times n}\right], \\ \Theta^c &= (I + HBQ\bar{I})HW,\ \Xi^c = (I + HBQ\bar{I})A, \end{aligned} \qquad (15)$$

with A, B, H and W defined in Appendix A, which are obtained by stacking the system dynamics in (13), whereas μ^e is obtained from (12). Further, if (14) is feasible, we can find the controller gains for (5) via:

$$K = (I + Q\bar{I}HB)^{-1}Q,\ \lambda = (I + Q\bar{I}HB)^{-1}r, \qquad (16)$$

where $K_{k\div T_d,i}$ and $\lambda_{k\div T_d}$ can be obtained from K and λ as can be seen in their definitions in Appendix A.

Proof. Since the design is within a finite horizon of length T_d, we can stack the equations in (13) and (5) over $k \in \mathbb{Z}_0^{T_d}$ to obtain the following matrix forms:

$$\begin{aligned} e &= Ae_0 + HBu^c + HWw, \\ y^c &= \bar{I}e - \tilde{x},\ u^c = \lambda + Ky^c, \end{aligned} \qquad (17)$$

with stacked trajectories defined as:

$$\begin{aligned} e &= [e_0^\top, e_1^\top, \ldots, e_{T_d}^\top]^\top, w = [w_0^\top, w_1^\top, \ldots, w_{T_d-1}^\top]^\top, \\ y^c &= [(y_0^c)^\top, (y_1^c)^\top, \ldots, (y_{T_d-1}^c)^\top]^\top, \\ \tilde{x} &= [\tilde{x}_0^\top, \tilde{x}_1^\top, \ldots, \tilde{x}_{T_d-1}^\top]^\top, \lambda = [\lambda_0^\top, \lambda_1^\top, \ldots, \lambda_{T_d-1}^\top]^\top. \\ u^c &= [(u_0^c)^\top, (u_1^c)^\top, \ldots, (u_{T_d-1}^c)^\top]^\top. \end{aligned}$$

Then, following similar steps as the proof in Rutledge et al. (2018) for the perfect measurement case, we leverage Q-parameterization to design $Q = K(I - \bar{I}HBK)^{-1}$ and $r = (I + Q\bar{I}HB)\lambda$ to convert the bilinear problem into a linear one, which results in the relationship for e as follows:

$$e = \Theta^c w - HBQ\tilde{x} + \Xi^c e_0 + HBr.$$

The rest of the construction follows from the constraints required to satisfy equalized recovery, as defined in Definition 1. ■

The optimization problem (14) contains semi-infinite constraints, but can be converted to an equivalent problem with finite constraints by applying robust optimization techniques in Ben-Tal et al. (2009).

Further, as with the dynamic state estimator, the periodic nature of the delayed data model allows us to guarantee that the tracking (control) error satisfies equalized recovery (cf. Definition 1) in a periodic manner, i.e., the tracking (control) errors $e_k = x_k - \bar{x}_k$ for all $k \in \mathbb{Z}$ satisfy time-varying controller tubes, as follows:

$$e_k \in S_{k\div T_d}^c \triangleq \{x \mid \|x\| \le \mu_{k\div T_d}^c\}. \qquad (18)$$

3.1.3. Nominal MPC with Time-Varying Tubes: For the nominal system in (4) with tightened state and input constraints, where the tightening is based on (12) and (18), the following proposition provides the optimization problem for the nominal MPC design that is solved at each time step k to obtain the optimal nominal input $\bar{u}_k$ and state $\bar{x}_k$ as the first solutions $\bar{u}_k^{k*}$ and $\bar{x}_k^{k*}$ (cf. (8)).

Theorem 5. Suppose that the following nominal MPC problem with prediction horizon T_p satisfies recursive feasibility (cf. Theorem 6):

$$\min_{\bar{x}_i^k, \bar{u}_i^k} \sum_{i=k}^{k+T_p-1} (\bar{x}_i^{k\top} Q \bar{x}_i^k + \bar{u}_i^{k\top} R \bar{u}_i^k) + \bar{x}_{k+T_p}^{k\top} Q_f \bar{x}_{k+T_p}^k,$$

$$\text{s.t.} \quad \bar{x}_{i+1}^k = A\bar{x}_i^k + B\bar{u}_i^k,\ \bar{x}_{k+T_p}^k \in \mathcal{X}_f, \qquad (19a)$$

$$\bar{x}_i^k \in \mathcal{X} \ominus S_{i\div T_d}^c, \qquad (19b)$$

$$\lambda_i + \bar{u}_i^k \in \mathcal{U} \ominus \bigoplus_{j=0}^{i\div T_d} K_{(i\div T_d,j)}(S_j^c \oplus (-S_j^e)), \qquad (19c)$$

where $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ and $Q_f \in \mathbb{R}^{n\times n}$ are positive definite matrices corresponding to state, input and terminal costs/penalties, respectively. Then, for the system (1) and its nominal counterpart (4) with tightened state and input constraints based on (12) and (18), and with a terminal set $\mathcal{X}_f$, the solution to the above problem (19), along with the feedback law (9), the dynamic state estimator (2)–(3) and the tracking controller (5), guarantees that the system (1) always satisfies the (original) state and input constraints, $\mathcal{X}$ and $\mathcal{U}$.

Proof. From the optimization problem (19), it can be seen that the nominal state at each time step k always satisfies the tightened state constraint $\bar{x}_k \in \mathcal{X} \ominus S_{k\div T_d}^c$, while from (18), we have $e_k = x_k - \bar{x}_k \in S_{k\div T_d}^c$. This implies that $x_k \in \bar{x}_k \oplus S_{k\div T_d}^c$. Since $\bar{x}_k \in \mathcal{X} \ominus S_{k\div T_d}^c$, then $x_k \in \mathcal{X} \ominus S_{k\div T_d}^c \oplus S_{k\div T_d}^c \Rightarrow x_k \in \mathcal{X}$, i.e., $x_k \in \mathcal{X}$ for all

$k \in \mathbb{Z}$. Similarly, the input constraint in the optimization problem (19) enforces that the nominal input at each time step k satisfies the tightened constraint (7). Since $u_k = \bar{u}_k + u_k^c$ from (9) and $u_k^c = \sum_{i=0}^{k \div T_d} K_{(k \div T_d, i)}(\hat{x}_i - \bar{x}_i) + \lambda_{k \div T_d}$ from (5), we obtain $u_k \in \mathcal{U}$ for all $k \in \mathbb{Z}$. ■

It is noteworthy the sets S^e and S^c in (12) and (18) are hyperboxes; hence, the set constraints in (19b) and (19c) can be implemented by imposing those constraints for all of their vertices. Moreover, in the next section, we will derive some sufficient conditions on the terminal and stage cost functions and the terminal set in (19) to obtain recursive feasibility and robust exponential stability.

3.2 Recursive Feasibility and Robust Stability

To prove recursive feasibility of the proposed tube-based MPC approach and the robust exponential stability of the closed-loop system, we adopt some common assumptions on the terminal set constraint, cost functions, etc.

Assumption 1. There exist a dynamic state estimator and a tracking controller such that the estimation and tracking errors satisfy (12) and (18), i.e., the estimator design from Hassaan et al. (2021) and (14) in Lemma 4 are feasible.

Assumption 2. There exists a feedback control gain K_f for the terminal set $\mathcal{X}_f$ such that if $\bar{x}_k \in \mathcal{X}_f$, then $(A_k + B_k K_f)\bar{x}_k \in \mathcal{X}_f$, where $\mathcal{X}_f \subseteq \overline{\mathcal{X}} \triangleq \mathcal{X} \ominus \bigcup_{i=0}^{T_d-1} S_i^c$ and $K_f \mathcal{X}_f \subseteq \overline{\mathcal{U}} \triangleq \bigcap_{i=1}^{T_d-1} \mathcal{U}_i$ with $\mathcal{U}_i \triangleq \mathcal{U} \ominus (\{\lambda_i\} \oplus \bigoplus_{j=0}^{T_d-1} K_{(i,j)}(S_j^c \oplus (-S_j^e)))$. Moreover, $\overline{\mathcal{X}}$ and $\overline{\mathcal{U}}$ are nonempty.

Assumption 3. The terminal cost function $V_f(\bar{x}_k) = \bar{x}_k^\top Q_f \bar{x}_k$ and the stage cost function $\ell(\bar{x}_k, \bar{u}_k) = \bar{x}_k^\top Q \bar{x}_k + \bar{u}_k^\top R \bar{u}_k$ in (19) satisfy

$$V_f((A_k + B_k K_f)\bar{x}_k) + l(\bar{x}_k, K_f \bar{x}_k) \le V_f(\bar{x}_k), \ \forall \bar{x}_k \in \mathcal{X}_f$$

with K_f and $\mathcal{X}_f$ satisfying Assumption 2.

Theorem 6. (Recursive Feasibility). If Assumption 2 holds and the optimization problem in (19) is feasible at $k = 0$, then (19) remains feasible at all times $k \in \mathbb{Z}$.

Proof. We will prove this by induction. The base case holds by the assumption that (19) is feasible at $k = 0$. For the inductive step, suppose that (19) is feasible at any predictive time step k. Then, there exist sequences of nominal states and inputs, i.e., $\{\bar{x}_i^{k*}\}_{i=k}^{k+T_p-1}$ and $\{\bar{u}_i^{k*}\}_{i=k}^{k+T_p-1}$, that satisfy the tightened constraints in (19b) and (19b), and $\bar{x}_{k+T_p}^{k*} \in \mathcal{X}_f$. At the following predictive time step $k+1$, the sequences $\{\bar{x}_i^{(k+1)*}\}_{i=k+1}^{k+T_p-1} = \{\bar{x}_i^{k*}\}_{i=k+1}^{k+T_p-1}$ and $\{\bar{u}_i^{(k+1)*}\}_{i=k+1}^{k+T_p-1} = \{\bar{u}_i^{k*}\}_{i=k+1}^{k+T_p-1}$ still satisfy the tightened constraints in (19b) and (19c), while Assumption 2 guarantees that $\bar{x}_{k+T_p+1}^{(k+1)*}$ satisfies the terminal condition in (19a), as well as that $\bar{x}_{k+T_p}^{(k+1)*} = \bar{x}_{k+T_p}^{k*}$ and $\bar{u}_{k+T_p}^{(k+1)*} = \bar{u}_{k+T_p}^{k*}$ satisfy (19b) and (19c), respectively, since $\bar{x}_{k+T_p}^{k*} \in \mathcal{X}_f \subseteq \overline{\mathcal{X}} \subseteq \mathcal{X} \ominus S_{(k+T_p)\div T_d}^c$ and $\bar{u}_{k+T_p}^{k*} = K_f \bar{x}_{k+T_p}^{k*} \in \overline{\mathcal{U}} \subseteq \mathcal{U}_{(k+T_p)\div T_d}$. Hence, recursive feasibility holds. ■

Theorem 7. (Robust Exponential Stability). Suppose the Assumptions 1, 2 and 3 hold and let $\mathbb{X} \triangleq \{\bar{x}_0 \mid$ (19) with $k = 0$ is feasible$\}$ be bounded and $\mathbb{S} \triangleq \bigcup_{j=0}^{T_d-1} S_i^c = \{x \mid \|x\| \le \max_{j=0}^{T_d-1} \mu_j^c\}$. Then, for any initial states $x_0 \in \mathcal{X}$, $\bar{x}_0 \in \mathbb{X}$, $\tilde{x}_0 \in S_0^e$ and $e_0 \in S_0^c$, the system state x_k is robustly steered to $\mathbb{S}$ exponentially fast (i.e., the set $\mathbb{S}$ is robustly exponentially stable for the system in (1); cf. Definition 2) while satisfying the state and input constraints, $\mathcal{X}$ and $\mathcal{U}$.

Proof. Under Assumptions 2 and 3 and the boundedness of $\mathbb{X}$, it was shown in Mayne et al. (2005) that the value function of the optimization problem in (19) has a nice descent property and as a result, the MPC control law based on (19) exponentially stabilizes $\bar{x}_k$ to the origin. Moreover, by design (cf. Assumption 1), the dynamic state estimator and tracking controller in Sections 3.1.1 and 3.1.2 guarantee that $e_k = x_k - \bar{x}_k \in S_{k\div T_d}^c \subseteq \mathbb{S}$. Putting these observations together results in the robust exponential stability of $\mathbb{S}$ for system (1), since $x_k = e_k - \bar{x}_k$ is robustly steered to $\mathbb{S}$ exponentially fast. ■

4. SIMULATION EXAMPLE

In this section, we demonstrate the effectiveness of our proposed tube-based output feedback MPC design in the presence of delayed/missing data via an example with double integrator dynamics, similar to Mayne et al. (2006):

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w_k,$$
$$z_k = \begin{bmatrix} 1 & 1 \end{bmatrix} x_k + v_k,$$

where $x_k \in \mathcal{X} \triangleq [-50, 3] \times [-50, 3]$, $u_k \in \mathcal{U} \triangleq [-3, 3]$, $w_k \in \mathcal{W} \triangleq [-0.05, 0.05] \times [-0.05, 0.05]$ and $v_k \in \mathcal{V} = [-0.01, 0.01]$. Further, we consider a delayed data model with a period of $T_d = 5$, and within each period, the plant output at the 1st, 2nd and 3rd steps are either on time or delayed by 1 step. Moreover, the outputs at 2nd and 3rd steps can also be missing. These delayed and missing data patterns are captured in a finite-length delayed data language of period $T_d = 5$ as follows:

$$\mathcal{L} = \left\{ \begin{array}{l} 00000,\ 00100,\ 00500,\ 01000,\ 01100,\ 01500, \\ 00010,\ 00110,\ 00510,\ 01010,\ 01110,\ 01510, \\ 00050,\ 00150,\ 00550,\ 01050,\ 01150,\ 01550 \end{array} \right\}.$$

Using the estimator design from Hassaan et al. (2021), we obtain the path-dependent estimator gains $(M^\alpha, L^\alpha, \nu^\alpha)$ and the recovery levels in (12) as $\{\mu_i^e\}_{i=0}^{T_d-1} = \{0.157, 0.207, 0.317, 0.23, 0.45\}$. On the other hand, the tracking controller design in Lemma 4 yields controller gains (K, λ) and recovery levels in (18) as $\{\mu_i^c\}_{i=0}^{T_d-1} = \{0.645, 0.6827, 0.7022, 0.7256, 0.8135\}$. Moreover, we utilize a terminal set $\mathcal{X}_f$ that satisfies Assumption 2 and $Q = I$, $R = 0.01I$ in the nominal MPC problem in (19) with $T_p = 13$, which is recursively solved with the true delayed data pattern as 0100001550011000..., where 0 means that the data was on time, 1 means a delay of 1 time step, and 5 means that the data was missing, and with initial nominal state $\bar{x}_0 = [-3, -8]^\top$, state x_0 chosen randomly within $x_0^n \oplus S_0^c$ and estimate $\hat{x}_0$ chosen randomly within $x_0 \oplus S_0^e$.

The result of the simulation is shown in Figure 1, where the state x_k is robustly steered into $\mathbb{S}$ (in green) exponentially fast, while remaining in time-varying estimator and controller tubes (in magenta and blue) and satisfying the state and input constraints $\mathcal{X}$ and $\mathcal{U}$. When compared to the trajectory depicted in (Mayne et al., 2006, Figure 1) where there was no missing/delayed data, the solution of our proposed tube-based output feedback MPC takes a longer trajectory but is able to robustly control the system to $\mathbb{S}$ even when some data are delayed and/or missing.

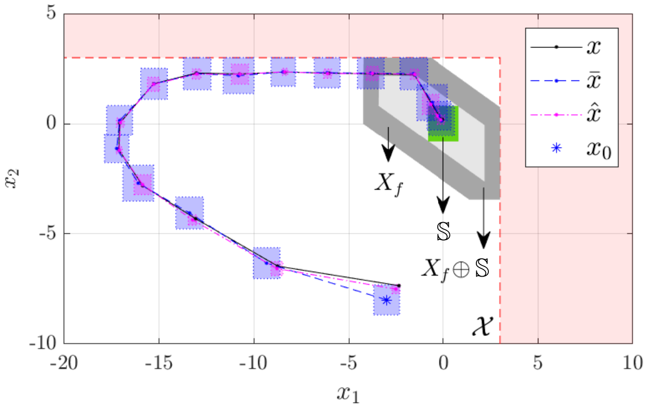

Fig. 1. Trajectories of the actual states, nominal states and state estimates with the proposed tube-based output feedback MPC, where x_k is robustly steered to $\mathbb{S}$ (in green). The blue boxes at each step represent the controller tube formed from $S^c_{k\div T_d}$ and the magenta boxes represent the estimator tube from $S^e_{k\div T_d}$.

5. CONCLUSIONS

In this paper, we proposed a time-varying tube-based output feedback MPC for constrained linear systems in the presence of missing and delayed data whose patterns satisfy a periodic finite-length language. The design consists of a dynamic state estimator with time-varying finite estimation error bounds and an output feedback tube-based MPC design that extends existing tube-based output feedback MPC methods to permit time-varying tubes for tightening the state and input constraints. Moreover, we derived sufficient conditions for recursive feasibility and robust exponential stability of the proposed design. The effectiveness of our proposed method to stabilize a constrained linear system despite intermittently delayed observations was demonstrated via a simulation example.

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Appendix A. MATRIX DEFINITIONS

Matrices and vectors used in Lemma 4 and those obtained from Hassaan et al. (2021) are defined as follows:

$$A=\begin{bmatrix} I_n \\ A_0^1 \\ \vdots \\ A_0^{T_d} \end{bmatrix}, B=\begin{bmatrix} B_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_{T_d-1} \end{bmatrix}, W=\begin{bmatrix} W_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & W_{T_d-1} \end{bmatrix},$$

$$L^\alpha=\begin{bmatrix} L_0^\alpha & 0 & \cdots & 0 \\ 0 & L_1^\alpha & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & L_{T_d-1}^\alpha \end{bmatrix}, H=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ A_1^1 & 0 & 0 & \cdots & 0 \\ A_1^2 & A_2^2 & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ A_1^{T_d} & A_2^{T_d} & A_3^{T_d} & \cdots & A_{T_d}^{T_d} \end{bmatrix},$$

$$\nu^\alpha=\begin{bmatrix} \nu_0^\alpha \\ \nu_1^\alpha \\ \vdots \\ \nu_{T_d-1}^\alpha \end{bmatrix}, M^\alpha=\begin{bmatrix} M^\alpha_{(0,0)} & 0 & \cdots & 0 \\ M^\alpha_{(1,0)} & M^\alpha_{(1,1)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ M^\alpha_{(T_d-1,0)} & M^\alpha_{(T_d-1,1)} & \cdots & M^\alpha_{(T_d-1,T_d-1)} \end{bmatrix},$$

$$K=\begin{bmatrix} K_{(0,0)} & 0 & \cdots & 0 \\ K_{(1,0)} & K_{(1,1)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ K_{(T_d-1,0)} & K_{(T_d-1,1)} & \cdots & K_{(T_d-1,T_d-1)} \end{bmatrix}, \lambda=\begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{T_d-1} \end{bmatrix},$$

for all $\alpha \in \mathbb{N}_1^{|\mathcal{L}^{E'}|}$, where $A_i^k = A_{k-1}A_{k-2}...A_i$.