Partition-Based Parametric Active Model Discrimination Design with Applications to Driver Intention Estimation

Ruochen Niu, Qiang Shen and Sze Zheng Yong

Abstract— **In this paper, we propose a partition-based parametric active model discrimination approach for designing optimal input sequences that distinguish a set of discrete-time affine time-invariant models with uncontrolled inputs, model-independent parameters and noises over a fixed horizon, where the parameters are revealed in real-time. By partitioning the operating region of the parameters, the input design problem is formulated as a sequence of offline optimization problems. Thus, at each time instant, one only needs to determine which subregions in the resulting partition tree that the revealed parameters lie in and select the corresponding pre-computed inputs. The offline optimal input design problem is formulated as a bilevel problem and further cast as a mixed-integer linear program. Finally, we demonstrate the effectiveness of the proposed active model discrimination approach for identifying the intention of other vehicles in a lane changing scenario.**

I. Introduction

Cyber-physical systems (CPS), such as the power grid, autonomous driving, aerospace system, are ubiquitous and have a big influence on our daily life. However, the often complex CPS are inevitably interconnected with other systems, whose models and behavior patterns such as intention, are not accessible or only partially known, making it nontrivial to provide safety guarantees. For example, autonomous vehicles operate and make decisions such as collision avoidance without the knowledge of the intentions of surrounding drivers or pedestrians [1], [2]. Similarly, it is of great interest to rapidly and accurately determine from noisy measurements whether a system malfunction or faults have occurred and which components have failed. These are broadly studied in the field of statistics, machine learning and systems theory.

Literature Review: Approaches for model discrimination, including automated fault diagnosis and intention identification, can be categorized into passive and active methods. Passive approaches, which are more broadly studied, compare collected input-output data in real time with existing data to separate models regardless of the input [3], [4], [5]. By contrast, active methods, also known as active model discrimination, exert a minimal signal or input into the system to ensure that the behaviors of all models are distinct and discriminated from each other [6], [7], [8], [9], using various techniques ranging from polyhedral projection [7] to a mixed-integer linear program (MILP) [8], [9].

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Furthermore, closed-loop approaches that consider online measurements have been studied in [10], [11] in order to reduce the input cost and time needed for model discrimination. In [11], a multi-parametric method that moves most of the computation offline is proposed, while in [10], a partition-based method, similar to the one we are proposing, is considered. Both approaches [10], [11] use an explicit set representation of the reachable states in the form of zonotopes and additionally require the design of a set-valued observer in a moving horizon fashion. To our knowledge, this approach does not directly apply when there are time-varying parameters that are revealed in real time, as in multistage optimization problems, e.g., [12], and when the time horizon is fixed, as is the case we are addressing.

Contributions: This paper presents a partition-based parametric approach for active model discrimination amongst a set of discrete-time affine time-invariant models with common model-independent parameters that are only known or revealed in real time. These time-varying parameters can represent real-time information such as varying road gradients or reference outputs. By leveraging the additional information from the revealed parameters, the "adaptive" separating input will lead to improved performance (i.e., lower input cost) when compared to an open-loop approach that does not take this information into account (cf. [9], [13]). Since solving the active model discrimination problem in real time is computationally demanding, we propose to move this active model discrimination problem offline, by solving them as a function of the parameters (as parametric variables), i.e., as a parametric active model discrimination problem. To further alleviate the computational burden, we consider partitions of the operating regions of the parameters.

We formulate the parametric active model discrimination problem as a sequence of offline optimization problems, each of which can be cast as a bilevel problem and converted to a tractable mixed-integer linear program (MILP) with Special Ordered Set of degree 1 (SOS-1) constraints. Comparing with existing approaches [7], [10], our approach uses an implicit set representation of the states, in contrast to the rather limiting explicit set representation, e.g., polyhedrons and zonotopes in [7], [10], respectively. Moreover, our formulation applies to a more general class of affine models with uncontrolled inputs and model-independent parameters beyond the classes of models considered in [7], [8], [10]. Finally, we illustrate the effectiveness of the proposed partition-based parametric active model discrimination approach to discriminate amongst the intentions of other human-driven or autonomous vehicles in a lane changing scenario.

II. Preliminaries

In this section, we introduce some notations and definitions, and describe the modeling framework we consider.

A. Notation and Definitions

Let $x \in \mathbb{R}^n$ denote a vector and $M \in \mathbb{R}^{n\times m}$ a matrix, with transpose $M^\intercal$ and $M \geq 0$ denotes element-wise non-negativity. The vector norm of x is denoted by $\|x\|_i$ with $i \in \{1, 2, \infty\}$, while $\mathbf{0}$, $\mathbb{1}$ and $\mathbb{I}$ represent the vector of zeros, the vector of ones and the identity matrix of appropriate dimensions. The diag and vec operators are defined for a collection of matrices $M_i, i = 1, \ldots, n$ and matrix M as:

$$\mathrm{diag}_{k=i}^{j}\{M_k\} = \begin{bmatrix} M_i & & \\ & \ddots & \\ & & M_j \end{bmatrix},\ \mathrm{vec}_{k=i}^{j}\{M_k\} = \begin{bmatrix} M_i \\ \vdots \\ M_j \end{bmatrix},$$
$$\mathrm{diag}_{i,j}\{M_k\} = \begin{bmatrix} M_i & \mathbf{0} \\ \mathbf{0} & M_j \end{bmatrix}, \qquad \mathrm{vec}_{i,j}\{M_k\} = \begin{bmatrix} M_i \\ M_j \end{bmatrix},$$
$$\mathrm{diag}_N\{M\} = \mathbb{I}_N \otimes M, \qquad \mathrm{vec}_N\{M\} = \mathbb{1}_N \otimes M,$$

where $\otimes$ is the Kronecker product.

The set of positive integers up to n is denoted by $\mathbb{Z}_n^+$, and the set of non-negative integers up to n is denoted by $\mathbb{Z}_n^0$. In addition, the set of non-negative integers from j_1 to j_2 $(0 \leq j_1 \leq j_2)$ is denoted by $\mathbb{Z}_{j_2}^{j_1}$.

Definition 1 (SOS-1 Constraints [14])**.** *A special ordered set of degree 1 (SOS-1) constraint*[1] *is a set of integer, continuous or mixed-integer scalar variables for which at most one variable in the set may take a value other than zero, denoted as SOS-1: $\{v_1, \ldots, v_N\}$. For instance, if $v_i \neq 0$, then this constraint imposes that $v_j = 0$ for all $j \neq i$.*

Definition 2 (Partition)**.** *A partition of a polyhedral set $\mathcal{P}$ is a collection of $|\mathcal{S}|$ disjoint subsets $\mathcal{P}_{\hat{p}}$ such that $\bigcup_{\hat{p}\in\mathcal{S}} \mathcal{P}_{\hat{p}} = \mathcal{P}$, where each partition $\mathcal{P}_{\hat{p}}$ is also a polyhedral set.*

B. Modeling Framework

Consider N discrete-time *parametric* affine time-invariant models $\mathcal{G}_i = (A_i, B_i, B_{w,i}, C_i, f_i, g_i)$, each with states $\vec{\boldsymbol{x}}_i \in \mathbb{R}^n$, outputs $z_i \in \mathbb{R}^{n_z}$, inputs $\vec{\boldsymbol{u}}_i \in \mathbb{R}^m$, process noise $w_i \in \mathbb{R}^{m_w}$, measurement noise $v_i \in \mathbb{R}^{m_v}$ and a common model-independent parameter $p(k) \in \mathbb{R}^{m_p}$ that is only known or revealed in real time. The models evolve according to the following state and output equations:

$$\begin{aligned} \vec{\boldsymbol{x}}_i(k+1) = A_i\vec{\boldsymbol{x}}_i(k) + B_i\vec{\boldsymbol{u}}_i(k) + B_{p,i}p(k) \\ + B_{w,i}w_i(k) + f_i, \end{aligned} \quad (1)$$
$$z_i(k) = C_i\vec{\boldsymbol{x}}_i(k) + v_i(k) + g_i. \quad (2)$$

The initial condition for model i, denoted by $\vec{\boldsymbol{x}}_i^0 = \vec{\boldsymbol{x}}_i(0)$, is constrained to a polyhedral set with c_0 inequalities:

$$\vec{\boldsymbol{x}}_i^0 \in \mathcal{X}_0 = \{\vec{\boldsymbol{x}} \in \mathbb{R}^n : P_0\vec{\boldsymbol{x}} \leq p_0\},\ \forall i \in \mathbb{Z}_N^+. \quad (3)$$

[1]Off-the-shelf solvers such as Gurobi and CPLEX [14], [15] can readily handle these constraints, which can significantly reduce the search space for integer variables in branch and bound algorithms.

The first m_u components of $\vec{\boldsymbol{u}}_i$ are controlled inputs (i.e., to be designed as separating inputs), denoted as $u \in \mathbb{R}^{m_u}$, which are the same for all $\vec{\boldsymbol{u}}_i$, while the other $m_d = m - m_u$ components of $\vec{\boldsymbol{u}}_i$, denoted as $d_i \in \mathbb{R}^{m_d}$, are uncontrolled inputs that are model-dependent. Further, the states $\vec{\boldsymbol{x}}_i$ are divided into $x_i \in \mathbb{R}^{n_x}$ and $y_i \in \mathbb{R}^{n_y}$, where $n_y = n - n_x$:

$$\vec{\boldsymbol{u}}_i(k) = \begin{bmatrix} u(k) \\ d_i(k) \end{bmatrix}, \vec{\boldsymbol{x}}_i(k) = \begin{bmatrix} x_i(k) \\ y_i(k) \end{bmatrix}. \quad (4)$$

The states x_i and y_i represent the subset of the states $\vec{\boldsymbol{x}}_i$ that are the 'responsibilities' of the controlled and uncontrolled inputs, which are to be interpreted as u and d_i with the following polyhedral domains (for $k \in \mathbb{Z}_{T-1}^0$) with c_u and c_d inequalities, respectively:

$$u(k) \in \mathcal{U} = \{u \in \mathbb{R}^{m_u} : Q_u u \leq q_u\}, \quad (5)$$
$$d_i(k) \in \mathcal{D}_i = \{d \in \mathbb{R}^{m_{d_i}} : Q_{d,i}d \leq q_{d,i}\}, \quad (6)$$

that must independently ensure the satisfaction of following polyhedral state constraints (for $k \in \mathbb{Z}_T^+$) with c_x and c_y inequalities:

$$x_i(k) \in \mathcal{X}_{x,i} = \{x \in \mathbb{R}^{n_x} : P_{x,i}x \leq p_{x,i}\}, \quad (7)$$
$$y_i(k) \in \mathcal{X}_{y,i} = \{y \in \mathbb{R}^{n_y} : P_{y,i}y \leq p_{y,i}\}. \quad (8)$$

On the other hand, the process noise w_i and measurement noise v_i are also polyhedrally constrained with c_w and c_v inequalities, respectively:

$$w_i(k) \in \mathcal{W}_i = \{w \in \mathbb{R}^{m_w} : Q_{w,i}w \leq q_{w,i}\}, \quad (9)$$
$$v_i(k) \in \mathcal{V}_i = \{v \in \mathbb{R}^{m_v} : Q_{v,i}v \leq q_{v,i}\}, \quad (10)$$

and have no responsibility to satisfy any state constraints.

Using the partitioned states and inputs, the corresponding state and output equations in (1) and (2) are rewritten as:

$$\begin{aligned} \vec{\boldsymbol{x}}_i(k+1) &= \begin{bmatrix} A_{xx,i} & A_{xy,i} \\ A_{yx,i} & A_{yy,i} \end{bmatrix} \vec{\boldsymbol{x}}_i(k) + \begin{bmatrix} B_{xu,i} & B_{xd,i} \\ B_{yu,i} & B_{yd,i} \end{bmatrix} \vec{\boldsymbol{u}}_i(k) \\ &+ \begin{bmatrix} B_{xp,i} \\ B_{yp,i} \end{bmatrix} p(k) + \begin{bmatrix} B_{xw,i} \\ B_{yw,i} \end{bmatrix} w_i(k) + \begin{bmatrix} f_{x,i} \\ f_{y,i} \end{bmatrix}, \end{aligned} \quad (11)$$
$$z_i(k) = C_i\vec{\boldsymbol{x}}_i(k) + v_i(k) + g_i. \quad (12)$$

Moreover, for the model-independent parameter $p(k)$ of each model $\mathcal{G}_i$, we assume that it has the following polyhedral operating region:

$$p(k) \in \mathcal{P} = \{p \in \mathbb{R}^{m_p} : Q_p p \leq q_p\}, \quad (13)$$

over the entire horizon T, i.e., $p(k) \in \mathcal{P}$, $\forall k \in \mathbb{Z}_{T-1}^0$. At current time step t, let $p_m(t)$ denote the revealed parameter, which implies that in real time, only $p_m(k)$, $\forall k \in \mathbb{Z}_t^0$ are available as a "feedback" term for active input design.

Note that the above modeling framework is an extension of [9], [13] with the inclusion of the revealed parameters $p_m(k)$, $\forall k \in \mathbb{Z}_t^0$ at current time t, as well as unrevealed parameters $p(k)$, $k > t$. The inclusion of the revealed parameters allows us to capture real-time information such as weight, road gradient, friction coefficients or reference outputs. The following example illustrates how reference outputs can be captured by (revealed/unrevealed) parameters:

Example 1. *Consider a simplified model given by*

$$x_i(k+1) = A_i^o x_i(k) + B_i u_i^o(k) + B_{w,i}w_i(k) + f_i,$$

with feedback control input $u_i^o(k)$ given by

$$u_i^o(k) = -K_i(y_i^o(k) - y^{des}(k)) + u(k), \qquad (14)$$

where K_i is a constant control gain matrix, $y_i^o(k) = C_i^o x_i(k)$ is the output and $y^{des}(k)$ denotes the model-independent desired time-varying output reference. Then, we can represent this model in the form of (1) *as:*

$$x_i(k+1) = A_i x_i(k) + B_i u(k) + B_{p,i} p(k) + B_{w,i} w_i(k) + f_i,$$

with $A_i = A_i^o - B_i K_i C_i^o$, $B_p = -B_i K_i$ and the model-independent parameter $p(k) = y^{des}(k)$.

Remark 1. *We assume throughout the paper that the given affine models are always* well-posed *[9], [13]. Otherwise, models are impractical.*

III. Problem Formulation

In this paper, we aim to design a causal separating input vector $u_T(p_{m,T-1})$ as a function of the revealed parameters $p_{m,T-1}$, whose k-th subvector is $u_k(p_{m,k}, \{u_{j-1}(\cdot)\}_{j=0}^k)$ with $p_{m,i} = \text{vec}_{k=0}^i\{p_m(k)\}$, $i \in \{0,\dots,T-1\}$ and $u_{-1}(\cdot) = \emptyset$. Specifically, $u_t(p_{m,t}, \{u_{j-1}(\cdot)\}_{j=0}^t))$ at each time instant $0 \le t \le T-1$ is computed by using only past and current revealed model-independent parameters $p_{m,t}$ as well as all past inputs $\{u_{j-1}(\cdot)\}_{j=0}^t$ (due to causality), so that the trajectories for all models are unique, while ensuring that the designed input is optimal with respect to a cost function $J(u_T^t)$. Since the separating input is updated with the newly available model-independent parameter p_m, the resulting performance will be improved when compared to an open-loop approach (see [9], [13] for example) that does not leverage this additional real-time information.

However, re-solving the open-loop optimization problem in real-time (i.e., at each time step) using the newly revealed parameters can be computationally prohibitive for many applications. Hence, we seek to alleviate the online computational burden by formulating it as a parametric active model discrimination problem that can be solved offline as a function of the revealed p_m as parametric variables. Moreover, to respect causality, we formulate the parametric active model discrimination problem as a sequence of optimization problems, where the input variable of the optimization problem at time instant t is constrained to inherit the optimal input subsequence $u_T^{*,t-1}(k)$ for all $k \in \mathbb{Z}_{t-1}^0$ from the previous time instant $t-1$ (see discussion in Section IV-A), as follows:

Problem 1 (Parametric Active Model Discrimination)**.** *Consider N well-posed affine models $\mathcal{G}_i$, and state, input and noise constraints defined in* (3) *and* (5)-(10)*. For each time instant $t \in \mathbb{Z}_{T-1}^0$ (sequentially, starting from $t=0$), with all past and current revealed $p_m(k) \in \mathcal{P}$, $\forall k \in \mathbb{Z}_t^0$ as parametric variables, and given the optimal input sequence $u_T^{*,t-1}$ from the previous time instant $t-1$, find an optimal input sequence $u_T^{*,t}$ with fixed horizon T to minimize a given cost function $J(u_T^t)$ subject to $u(k) = u_T^{*,t-1}(k)$ for all $k \in \mathbb{Z}_{t-1}^0$ such that for all possible initial states $\boldsymbol{x}_0$, uncontrolled inputs d_T, unrevealed parameters $p_{t+1:T-1}$, process noise w_T and measurement noise v_T, only one model is valid, i.e., the output trajectories of any pair of models have to differ in at least one time instant of the horizon T.*

Formally, for each time instant t with the newly revealed parameter $p_m(t)$ and all past parameters and chosen inputs, the optimization problem can be formulated as a sequence of optimization problems as follows (for $t = 0,\dots,T-1$):

$$u_T^{*,t} = \arg\min_{u_T, x_T} J(u_T)$$

$$\textit{s.t.} \quad \forall k \in \mathbb{Z}_{T-1}^0 : (5) \textit{ holds}, \qquad (15a)$$

$$\left.\begin{array}{r} \forall i,j \in \mathbb{Z}_{\mathrm{N}}^+, i<j, \forall k \in \mathbb{Z}_T^0, \\ \forall \boldsymbol{x}_0, y_T, d_T, p_{t+1:T-1}, w_T, v_T : \\ (1)\text{-}(3),\ (6),\ (8)\text{-}(10),\ (13)\ \textit{holds}; \\ \forall k \in \mathbb{Z}_t^0 : p(k) = p_m(k); \\ \forall k \in \mathbb{Z}_{t-1}^0 : u(k) = u_T^{*,t-1}(k) \end{array}\right\} : \begin{array}{l} \{\forall k \in \mathbb{Z}_T^+ : \\ (7)\ \textit{holds}\} \wedge \\ \{\exists k \in \mathbb{Z}_T^0 : \\ z_i(k) \neq z_j(k)\}. \end{array} \qquad (15b)$$

Note that the constraint (15b) not only enforces that the 'responsibility' of the controlled input is satisfied but also guarantees that all models are separated with the computed optimal input sequence $u_T^{*,t}$.

IV. Main Approach

In this section, we proposed a partition-based approach to solve the parametric active model discrimination problem defined in Problem 1. Although Problem 1 could be formulated and solved as a multi-parametric program, it remains computationally expensive when using currently available toolboxes, e.g., multi-parametric toolbox (MPT) [16], especially when there are many binary variables. Thus, in this paper, we consider a slightly more conservative approach that partitions the operating region $\mathcal{P}$ at each time instant, and solve a sequence of tractable optimization problems for each possible so that we can solve Problem 1 efficiently.

A. Partition

Recall the operating region $\mathcal{P}$ (cf. (13) for its definition) of the parameter p. At each time instant $t \in \mathbb{Z}_{T-1}^0$, let $\{\mathcal{P}_{\hat{p}_m(t)}(t)\}_{\hat{p}_m(t) \in \mathcal{S}(t)}$ be a *partition* of $\mathcal{P}$ (cf. Definition 2) with $\mathcal{S}(t) = \{1,\dots,|\mathcal{S}(t)|\}$, where $|\mathcal{S}(t)|$ may vary with time t. Each subregion $\mathcal{P}_{\hat{p}_m(t)}(t)$ is a polyhedral set:

$$\mathcal{P}_{\hat{p}_m(t)}(t) = \{p \in \mathbb{R}^{m_p} : Q_{\hat{p}_m(t)}(t) p \le q_{\hat{p}_m(t)}(t)\}. \qquad (16)$$

As a result, any newly revealed parameter $p_m(t) \in \mathcal{P}_{\hat{p}_m(t)}(t)$ at time instant $t \in \mathbb{Z}_{T-1}^0$ can be over-approximated by $p_m(t) = p(t)$ with $p(t) \in \mathcal{P}_{\hat{p}_m(t)}(t)$.

To clearly illustrate the partition-based method, an example partition tree for the parameter $p(t)$ over a fixed horizon $T=3$ is shown in Fig. 1. Since there is no revealed parameter before the initial time instant, i.e., $t=-1$, no partition is carried out at the very top of the partition tree. At time instant $t=0$, the operating region $\mathcal{P}$ is partitioned into two subregions $\mathcal{P}_1(0)$ and $\mathcal{P}_2(0)$ with $\mathcal{S}(0) = \{1,2\}$, and we pre-compute the separating inputs offline for the two subregions instead of performing online computation using the revealed parameter $p_m(0)$ itself. At the next time instant $t=1$, we partition the operating region $\mathcal{P}$ again into two subregions $\mathcal{P}_1(1)$ and $\mathcal{P}_2(1)$ with $\mathcal{S}(1) = \{1,2\}$, which need not be identical to $\mathcal{P}_1(0)$ and $\mathcal{P}_2(0)$ at $t=0$, respectively.

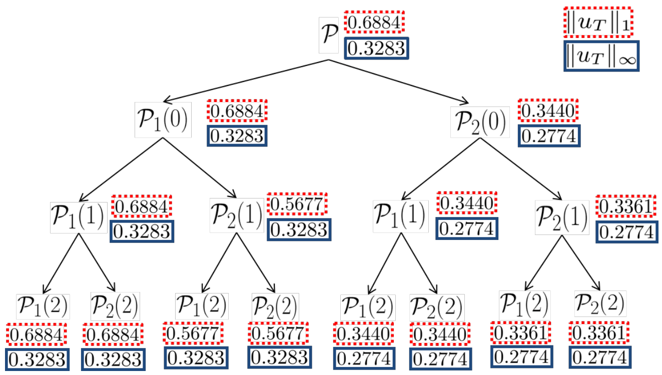


Fig. 1: Partition tree for the model-independent parameter p over a fixed horizon $T = 3$.

Proceeding with $t = 1$, we further partition the operating region into two for each of the preceding node and compute 4 optimal separating inputs corresponding to 4 different cases. Finally, at the last time instant $t = T - 1 = 2$, we again partition the operating regions into two subregions and compute all 8 optimal separating inputs, which correspond to 8 different cases/trajectories of subregions that the yet unknown revealed parameters $p_m(0)$, $p_m(1)$ and $p_m(2)$ may lie in. As a result, we simplify the online process of solving Problem 1 for each time instant $t \in \mathbb{Z}_{T-1}^0$ to a simple look-up operation according to the newly revealed parameter $p_m(t)$ and all past parameters. Moreover, to ensure causality, we constrain the inputs for each node (i.e., at time instant t) to inherit the inputs of all previous time instants (i.e., from 0 to $t-1$) from their parents. For instance, at the node corresponding the trajectory of $\{\mathcal{P}_2(0), \mathcal{P}_1(1), \mathcal{P}_1(2)\}$, the input u_T^2 is constrained such that $u_T^2(0)$ and $u_T^2(1)$ must be the same as the optimal input $u_T^1(0)$ and $u_T^1(1)$ from its parent with the trajectory $\{\mathcal{P}_2(0), \mathcal{P}_1(1)\}$.

The problem of the partition-based parametric active model discrimination can be formulated as follows:

Problem 2 (Partition-Based Parametric Active Model Discrimination)**.** *For each trajectory of the partition tree given by $\{\mathcal{P}_{\hat{p}_m(k)}(k)\}_{k=0}^{T-1}$, the parametric active model discrimination problem in Problem 1 can be reformulated as a sequence of optimization problems as follows (for $t = 0, \ldots, T-1$):*

$$u_T^{*,t} = \arg\min_{u_T, x_T} J(u_T)$$

$$\textit{s.t.} \quad \forall k \in \mathbb{Z}_{T-1}^0 : (5)\ \textit{holds}, \tag{17a}$$

$$\left.\begin{array}{r}\forall i,j \in \mathbb{Z}_N^+, i<j, \forall k \in \mathbb{Z}_T^0,\\ \forall \boldsymbol{x}_0, y_T, d_T, p_{t+1:T-1}, w_T, v_T :\\ (1)\text{-}(3),\ (6),\ (8)\text{-}(10),\ (13)\ \textit{hold}\\ \forall k \in \mathbb{Z}_t^0, \forall p(k) : p(k) \in \mathcal{P}_{\hat{p}_m(k)}(k);\\ \forall k \in \mathbb{Z}_{t-1}^0 : u(k) = u_T^{*,t-1}(k)\end{array}\right\} : \begin{array}{l}\{\forall k \in \mathbb{Z}_T^+ :\\ (7)\ \textit{holds}\} \wedge\\ \{\exists k \in \mathbb{Z}_T^0 :\\ z_i(k) \neq z_j(k)\},\end{array} \tag{17b}$$

where $u_T^{*,t-1}$ is the optimal input sequence from the previous time instant $t-1$. Note that any pair of trajectories that share the same node at time instant t on the partition tree, their optimal input subsequences up to t must be the same.

Comparing with Problem 1, the equality constraints on parameter $p(k)$ for all $k \in \mathbb{Z}_t^0$ are replaced by its over-approximation in Problem 2, which corresponds to the sub-region that the revealed parameter $p_m(k)$ lies in.

B. Time-Concatenated Model

Before proceeding with the main approach for solving the partition-based parametric active model discrimination problem, we introduce some time-concatenated notations and write the considered N models in a time-concatenated form. The time-concatenated states and outputs are defined as

$$\vec{\boldsymbol{x}}_{i,T} = \text{vec}_{k=0}^T\{\vec{\boldsymbol{x}}_i(k)\}, \quad x_{i,T} = \text{vec}_{k=0}^T\{x_i(k)\},$$
$$y_{i,T} = \text{vec}_{k=0}^T\{y_i(k)\}, \quad z_{i,T} = \text{vec}_{k=0}^T\{z_i(k)\},$$

while the time-concatenated inputs and noises are defined as

$$\vec{\boldsymbol{u}}_{i,T} = \text{vec}_{k=0}^{T-1}\{\vec{\boldsymbol{u}}_i(k)\}, u_T = \text{vec}_{k=0}^{T-1}\{u(k)\}, d_{i,T} = \text{vec}_{k=0}^{T-1}\{d_i(k)\},$$
$$w_{i,T} = \text{vec}_{k=0}^{T-1}\{w_i(k)\}, v_{i,T} = \text{vec}_{k=0}^{T}\{v_i(k)\}.$$

In addition, for $0 \le j_1 \le j_2 \le T-1$, we also define the model-independent parameter sequence as $p_{j_1:j_2} = [p(j_1)^\intercal, \ldots, p(j_2)^\intercal]^\intercal \in \mathbb{R}^{(j_2-j_1+1)m_p}$. Therefore, at each time instant $t \in \mathbb{Z}_{T-1}^0$ with the past and current revealed parameters $\{p_m(k)\}_{k=0}^t$ lying in subregions $\{\mathcal{P}_{\hat{p}_m(k)}(k)\}_{k=0}^t$, we have revealed parameters $p_{0:t} = \text{vec}_{k=0}^t\{p(k)\}$, where $p(k) \in \mathcal{P}_{\hat{p}_m(k)}(k), \forall k \in \mathbb{Z}_t^0$, and unrevealed parameters $p_{t+1:T-1} = \text{vec}_{k=t+1}^{T-1}\{p(k)\}$, where $p(k) \in \mathcal{P}, \forall k \in \mathbb{Z}_{T-1}^{t+1}$. The time-concatenated parameter vector over the entire horizon is further defined as $p_T = \left[p_{0:t}^\intercal \;\; p_{t+1:T-1}^\intercal\right]^\intercal$.

Given N discrete-time affine models, there are $I = \binom{N}{2}$ model pairs and let the mode $\iota \in \{1, \cdots, I\}$ denote the pair of models (i,j). Then, concatenating $\vec{\boldsymbol{x}}_i^0$, $\vec{\boldsymbol{x}}_{i,T}$, $x_{i,T}$, $y_{i,T}$, $d_{i,T}$, $z_{i,T}$, $w_{i,T}$ and $v_{i,T}$ for each model pair, we define

$$\vec{\boldsymbol{x}}_0^\iota = \text{vec}_{i,j}\{\vec{\boldsymbol{x}}_i^0\}, \;\; \vec{\boldsymbol{x}}_T^\iota = \text{vec}_{i,j}\{\vec{\boldsymbol{x}}_{i,T}\}, \;\; \vec{\boldsymbol{u}}_T^\iota = [u_T^\intercal, d_T^{\iota\intercal}]^\intercal,$$
$$x_T^\iota = \text{vec}_{i,j}\{x_{i,T}\}, \; y_T^\iota = \text{vec}_{i,j}\{y_{i,T}\}, \;\; z_T^\iota = \text{vec}_{i,j}\{z_{i,T}\},$$
$$d_T^\iota = \text{vec}_{i,j}\{d_{i,T}\}, \;\; w_T^\iota = \text{vec}_{i,j}\{w_{i,T}\}, \; v_T^\iota = \text{vec}_{i,j}\{v_{i,T}\}.$$

The states and outputs over the entire time horizon for each mode ι can be written as simple functions of the initial state $\vec{\boldsymbol{x}}_0^\iota$, inputs u_T, d_T^ι, parameter p_T and noises w_T^ι, v_T^ι:

$$x_T^\iota = M_x^\iota \vec{\boldsymbol{x}}_0^\iota + \Gamma_{xu}^\iota u_T + \Gamma_{xd}^\iota d_T^\iota + \Gamma_{xp}^\iota p_T + \Gamma_{xw}^\iota w_T^\iota + \tilde{f}_x^\iota, \tag{18}$$
$$y_T^\iota = M_y^\iota \vec{\boldsymbol{x}}_0^\iota + \Gamma_{yu}^\iota u_T + \Gamma_{yd}^\iota d_T^\iota + \Gamma_{yp}^\iota p_T + \Gamma_{yw}^\iota w_T^\iota + \tilde{f}_y^\iota, \tag{19}$$
$$\vec{\boldsymbol{x}}_T^\iota = \bar{A}^\iota \vec{\boldsymbol{x}}_0^\iota + \Gamma_u^\iota u_T + \Gamma_d^\iota d_T^\iota + \Gamma_p^\iota p_T + \Gamma_w^\iota w_T^\iota + \tilde{f}^\iota, \tag{20}$$
$$z_T^\iota = \bar{C}^\iota \vec{\boldsymbol{x}}_T^\iota + \bar{D}_d^\iota d_T^\iota + \bar{D}_v^\iota v_T^\iota + \tilde{g}^\iota. \tag{21}$$

The matrices and vectors $M_\star^\iota$, $\Gamma_{\star u}^\iota$, $\Gamma_{\star d}^\iota$, $\Gamma_{\star p}^\iota$, $\Gamma_{\star w}^\iota$ and $\tilde{f}_\star^\iota$ for $\star \in \{x, y\}$, and $\bar{A}^\iota$, Γ_u^ι, Γ_d^ι, Γ_p^ι, Γ_ω^ι, $\bar{C}^\iota$, $\bar{D}_d^\iota$, $\bar{D}_v^\iota$, $\tilde{f}\iota$, $\tilde{g}^\iota$ are defined in the appendix. Moreover, the uncertain variables for each mode ι are concatenated as $\bar{x}^\iota = [\vec{\boldsymbol{x}}_0^{\iota\intercal} \; d_T^{\iota\intercal} \; p_T^\intercal \; w_T^{\iota\intercal} \; v_T^{\iota\intercal}]^\intercal$.

We then concatenate the polyhedral state constraints in (7) and (8), eliminating x_T and y_T in them and expressing them in terms of $\bar{x}^\iota$ and u_T. First, let

$$\bar{P}_x^\iota = \text{diag}_{i,j}\,\text{diag}_T\{P_{x,i}\}, \quad \bar{P}_y^\iota = \text{diag}_{i,j}\,\text{diag}_T\{P_{y,i}\},$$
$$\bar{p}_x^\iota = \text{vec}_{i,j}\,\text{vec}_T\{p_{x,i}\}, \quad \bar{p}_y^\iota = \text{vec}_{i,j}\,\text{vec}_T\{p_{y,i}\}.$$

Then, we can rewrite the polyhedral constraints as:

$$\bar{P}_\star^\iota x_T^\iota \le \bar{p}_\star^\iota \Leftrightarrow H_{\star,t}^\iota \bar{x}^\iota \le h_{\star,t}^\iota(u_T), \; \star \in \{x, y\}$$

where $H^\iota_\star = \bar{P}^\iota_\star \begin{bmatrix} M^\iota_\star & \Gamma^\iota_{\star d} & \Gamma^\iota_{\star p} & \Gamma^\iota_{\star w} & \mathbf{0} \end{bmatrix}$ and $h^\iota_\star(u_T) = \bar{p}^\iota_\star - \bar{P}^\iota_\star \Gamma^\iota_{\star u} u_T - \bar{P}^\iota_\star \tilde{f}^\iota_\star$. Similarly, let

$$\bar{Q}_u = \mathrm{diag}_T\{Q_u\}, \ \ \bar{Q}^\iota_\dagger = \mathrm{diag}_{i,j}\,\mathrm{diag}_T\{Q_{\dagger,i}\},$$
$$\bar{q}_u = \mathrm{vec}_T\{q_u\}, \ \ \bar{q}^\iota_\dagger = \mathrm{vec}_{i,j}\,\mathrm{vec}_T\{q_{\dagger,i}\}, \ \dagger \in \{d, w, v\}.$$

Then, the uncertainties and input constraints in (5)-(6) and (9)-(10) over the entire horizon are equivalent to $\bar{Q}_u u_T \le \bar{q}_u$ and $\bar{Q}^\iota_\dagger \dagger^\iota_T \le \bar{q}^\iota_\dagger$. In addition, since the revealed parameters $\{p_m(k)\}_{k=0}^t$ are located in subregions $\{\mathcal{P}_{\hat{p}_m(k)}(k)\}_{k=0}^t$, the parameters $p_{0:t}$ satisfy

$$\bar{Q}_{p_m,0:t} p_{0:t} \le \bar{q}_{p_m,0:t},$$

where $\bar{Q}_{p_m,0:t} = \mathrm{diag}^t_{k=0}\{Q_{\hat{p}_m(k)}(k)\}$ and $\bar{q}_{p_m,0:t} = \mathrm{vec}^t_{k=0}\{q_{\hat{p}_m(k)}(k)\}$. Due to the fact that parameters $p_{t+1:T-1}$ are unrevealed at the time instant t, we also have

$$\bar{Q}_{p,t+1:T-1} p_{t+1:T-1} \le \bar{q}_{p,t+1:T-1},$$

where $\bar{Q}_{p,t+1:T-1} = \mathrm{diag}_{T-t-1}\{Q_p\}$ and $\bar{q}_{p,t+1:T-1} = \mathrm{vec}_{T-t-1}\{q_p\}$. As a consequence, the polyhedral constraint on p_T is obtained as

$$\bar{Q}_{p,t} p_T \le \bar{q}_{p,t},$$

with $\bar{Q}_{p,t} = \begin{bmatrix} \bar{Q}_{p_m,0:t} & 0 \\ 0 & \bar{Q}_{p,t+1:T-1} \end{bmatrix}$ and $\bar{q}_{p,t} = \begin{bmatrix} \bar{q}_{p_m,0:t} \\ \bar{q}_{p,t+1,T-1} \end{bmatrix}$.

Moreover, we concatenate the initial state constraint in (3):

$$\bar{P}^\iota_0 = \mathrm{diag}_2\{P_0\}, \quad \bar{p}^\iota_0 = \mathrm{vec}_2\{p_0\}.$$

Hence, in terms of $\bar{x}^\iota$, we have a polyhedral constraint of the form $H^\iota_{\bar{x},t} \bar{x}^\iota \le h^\iota_{\bar{x},t}$ for each time $t \in \mathbb{Z}^0_{T-1}$, with

$$H^\iota_{\bar{x},t} = \begin{bmatrix} \bar{P}^\iota_0 & 0 & 0 & 0 & 0 \\ 0 & \bar{Q}^\iota_d & 0 & 0 & 0 \\ 0 & 0 & \bar{Q}_{p,t} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}^\iota_w & 0 \\ 0 & 0 & 0 & 0 & \bar{Q}^\iota_v \end{bmatrix}, h^\iota_{\bar{x},t} = \begin{bmatrix} \bar{p}^\iota_0 \\ \bar{q}^\iota_d \\ \bar{q}_{p,t} \\ \bar{q}^\iota_w \\ \bar{q}^\iota_v \end{bmatrix}.$$

C. Active Model Discrimination Approach

Using the time-concatenated models, we proposed an optimization-based approach to solve Problem 2. Throughout this paper, we assume that the following holds:

Assumption 1. *In the concatenated constraint of the 'responsibility' of the uncontrolled input, i.e.,* $H^\iota_y \bar{x}^\iota \le \bar{p}^\iota_y - \bar{P}^\iota_y \tilde{f}^\iota_y - \bar{P}^\iota_y \Gamma^\iota_{yu} u_T$, $\bar{P}^\iota_y \Gamma_{yu} = 0$ *is satisfied.*

Note that Assumption 1 ensures that the resulting optimization problem does not have bilinear terms. If Assumption 1 does not hold, the problem (P_{PPDID}) results in a mixed-integer nonlinear program (MINLP). A particular solution to this problem is provided in [13], where a sequence of restriction approach reduces this MINLP into a computationally tractable sequence of optimization problems.

Due to the semi-infinite non-convex constraint (17b), Problem 2 is still not computationally tractable. To tackle this, we reformulate the optimization problem for each trajectory of the partition tree in Problem 2 as a bilevel optimization problem in the following lemma, and then further cast it into a single level optimization problem by applying KKT conditions in Theorem 1.

Lemma 1 (Bilevel Optimization Formulation)**.** *For each time instant* $t \in \mathbb{Z}^0_{T-1}$*, given a separability index* ϵ *and a trajectory on the partition tree corresponding to subregions* $\{\mathcal{P}_{\hat{p}_m(k)}(k)\}_{k=0}^{T-1}$*, the partition-based parametric active model discrimination problem in Problem 2 is equivalent to a sequence of bilevel optimization problems (for* $t = 0, \dots, T-1$*) with the following outer problem:*

$$u_T^{*,t} = \arg\min_{u_T} J(u_T) \qquad (P_{Outer})$$

$$\text{s.t.} \quad \forall i \in \mathbb{Z}^+_N, \forall k \in \mathbb{Z}^0_{T-1} : (5) \text{ holds}, \tag{22a}$$

$$\left.\begin{array}{r} \forall i,j \in \mathbb{Z}^+_{\mathrm{N}}, i<j, \forall k \in \mathbb{Z}^0_T, \\ \forall \boldsymbol{x}_0, y_T, d_T, p_{t+1:T-1}, w_T, v_T : \\ (1)\text{-}(3),\ (6),\ (8)\text{-}(10),\ (13)\ \textit{hold} \\ \forall k \in \mathbb{Z}^0_t, \forall p(k) : p(k) \in \mathcal{P}_{\hat{p}_m(k)}(k); \\ \forall k \in \mathbb{Z}^0_{t-1} : u(k) = u_T^{*,t-1}(k) \end{array}\right\} : \begin{array}{l} \forall k \in \mathbb{Z}^+_T : \\ (7)\ \textit{holds}, \end{array} \tag{22b}$$

$$\forall \iota \in \mathbb{Z}^+_{\mathrm{I}} : \delta^{\iota *}(u_T) \ge \epsilon, \tag{22c}$$

where $u_T^{*,t-1}$ *is the optimal input sequence from time instant* $t-1$ *and* $\delta^{\iota *}(u_T)$ *is the solution to the inner problem:*

$$\delta^{\iota *}(u_T) = \min_{\delta^\iota, \boldsymbol{x}^\iota_0, d^\iota_T, p_T, w^\iota_T, v^\iota_T} \delta^\iota \qquad (P_{Inner})$$

$$\text{s.t.} \quad \forall i \in \mathbb{Z}^+_N, \forall k \in \mathbb{Z}^0_{T-1} : (1) \text{ holds}, \tag{23a}$$

$$\forall i \in \mathbb{Z}^+_N, \forall k \in \mathbb{Z}^+_T : (2) \text{ holds}, \tag{23b}$$

$$\left.\begin{array}{r} \forall \boldsymbol{x}^\iota_0, y^\iota_T, d^\iota_T, \\ p_{t+1:T-1}, w^\iota_T, v^\iota_T \end{array}\right\} : \begin{array}{l} (3),(6),(13), \\ (8)\text{-}(10)\ \textit{hold}, \end{array} \tag{23c}$$

$$\forall k \in \mathbb{Z}^0_t, \forall p(k) : p(k) \in \mathcal{P}_{\hat{p}_m(k)}(k), \tag{23d}$$

$$\forall k \in \mathbb{Z}^0_{t-1} : u(k) = u_T^{*,t-1}(k), \tag{23e}$$

$$\forall l \in \mathbb{Z}^1_{n_z}, k \in \mathbb{Z}^0_T : |z_{i,l}(k) - z_{j,l}(k)| \le \delta^\iota. \tag{23f}$$

Proof. Since the universal quantifier distributes over conjunction [17, pp. 45–46], we can separate the constraint (17b) of Problem 2 into two independent constraints for all possible values of the uncertain variables at time instant $t \in \mathbb{Z}^0_{T-1}$, i.e., the 'responsibility' of the controlled input and the separation condition, respectively. Note that the constraint associated with the 'responsibility' of the controlled input is convex and kept in the outer problem. We only convert the non-convex separation condition by considering an equivalent formulation using double negation to get the inner problem. Details about implementing the double negation on the separation condition can be found in [9, Lemma 1]. □

Then, leveraging the literature on robust optimization [18], [19], we can convert the semi-infinite constraint in (22b) into linear constraints. Further, we can recast the bilevel optimization formulation in the foregoing lemma by applying KKT conditions to obtain an MILP with SOS-1 constraints, which can readily be solved using off-the-shelf optimization softwares, e.g., Gurobi and CPLEX [14], [15].

Theorem 1 (Partition-Based Parametric Discriminating Input Design as a Sequence of MILP)**.** *For each time instant* $t \in \mathbb{Z}^0_{T-1}$*, given a separability index* ϵ *and a trajectory on the partition tree corresponding to subregions* $\{\mathcal{P}_{\hat{p}_m(k)}(k)\}_{k=0}^{T-1}$*, the partition-based parametric active model discrimination problem (Problem 2) under Assumption 1 is equivalent to a* *sequence of MILP problems (for $t = 0, \dots, T-1$):*

$$u_T^{*,t} = \arg \min_{u_T, \Pi^\iota, \delta^\iota, \bar{x}^\iota, \mu_1^\iota, \mu_2^\iota, \mu_3^\iota} J(u_T) \qquad (P_{PPDID})$$

$$\begin{aligned}
&\textit{s.t.} \quad \bar{Q}_u u_T \leq \bar{q}_u,\\
&\forall k \in \mathbb{Z}_{t-1}^0 : \quad u(k) = u_T^{*,t-1}(k),\\
&\Pi^{\iota\mathsf{T}} \begin{bmatrix} h_{\bar{x},t}^\iota \\ \bar{p}_y^\iota - \bar{P}_y^\iota \tilde{f}_y^\iota \end{bmatrix} \leq \bar{p}_x^\iota - \bar{P}_x^\iota \tilde{f}_x^\iota - \bar{P}_x^\iota \Gamma_{xu}^\iota u_T,\\
&\Pi^\iota \operatorname{diag}\{H_{\bar{x},t}^\iota, H_y^\iota\} = H_x^\iota,\ \Pi^\iota \geq 0,\\
&\forall \iota \in \mathbb{Z}_I^+ : \ \delta^\iota(u_T) \geq \epsilon, 0 = 1 - \mu_3^{\iota\,T}\mathbb{1},\\
&0 = \textstyle\sum_{i=1}^{\kappa} \mu_{1,i}^\iota H_{\bar{x},t}^\iota(i,m) + \sum_{j=1}^{\xi} \mu_{2,j}^\iota R^\iota(j,m)\\
&\quad + \textstyle\sum_{j=\xi+1}^{\xi+\rho} \mu_{3,j-\xi}^\iota R^\iota(j,m), \forall m = 1, \cdots, \eta,\\
&\widetilde{H}_{\bar{x},t,i}^\iota \bar{x}^\iota - h_{\bar{x},t,i}^\iota \leq 0, \mu_{1,i}^\iota \geq 0, \forall i = 1, \dots \kappa,\\
&\widetilde{R}_j^\iota \bar{x}^\iota - r_j^\iota + S_j^\iota u_T \leq 0, \mu_{2,j}^\iota \geq 0, \forall j = 1, \dots \xi,\\
&\widetilde{R}_j^\iota \bar{x}^\iota - \delta^\iota - r_j^\iota + S_j^\iota u_T \leq 0, \mu_{3,j-\xi}^\iota \geq 0, \forall j = \xi+1, \dots \xi+\rho,\\
&\forall \iota \in \mathbb{Z}_I^+, \forall i \in \mathbb{Z}_\kappa^+ : \quad \textit{SOS-1} : \{\mu_{1,i}^\iota, \widetilde{H}_{\bar{x},t,i}^\iota \bar{x}^\iota - h_{\bar{x},t,i}^\iota\},\\
&\forall \iota \in \mathbb{Z}_I^+, \forall j \in \mathbb{Z}_\xi^+ : \quad \textit{SOS-1} : \{\mu_{2,j}^\iota, \widetilde{R}_j^\iota \bar{x}^\iota - r_j^\iota + \tilde{S}_j^\iota u_T\},\\
&\forall \iota \in \mathbb{Z}_I^+, \forall j \in \mathbb{Z}_{\xi+\rho}^{\xi+1} : \quad \textit{SOS-1} : \{\mu_{3,j-\xi}^\iota, \widetilde{R}_j^\iota \bar{x}^\iota - \delta^\iota - r_j^\iota + \tilde{S}_j^\iota u_T\},
\end{aligned}$$

where Π^ι, $\mu_{1,i}^\iota$, $\mu_{2,j}^\iota$, $\mu_{3,j-\xi}^\iota$ are dual variables, $\widetilde{H}_{\bar{x},t,i}^\iota$ is the i-th row of $H_{\bar{x},t}^\iota$, $\widetilde{R}_j^\iota$, $\widetilde{S}_j^\iota$ and r_j^ι are the j-th row of R^ι, S^ι and r^ι, $\eta = I(Tm_p + 2n + 2T(m_d + m_w + m_v))$ is the number of columns of $H_{\bar{x},t}^\iota$, $\kappa = I(Tm_p + 2c_0 + 2T(c_d + c_w + c_v))$ is the number of rows of $H_{\bar{x},t}^\iota$, $\xi = 2ITc_y$ is the number of rows of $\mathbf{0}$ in (30a), $\rho = 2ITn_z$ is the number of rows of $\mathbb{1}$ in (30a) and $u_T^{*,t-1}$ is the optimal input sequence from time instant $t-1$.

Proof. Since Assumption 1 holds, it can be verified that the concatenated form of (22b) is equivalently written as the following two constraints:

$$\forall k \in \mathbb{Z}_{t-1}^0 : \ u(k) = u_T^{*,t-1}(k), \tag{25}$$

$$\begin{bmatrix} \max_{\bar{x}^\iota} H_x^\iota \bar{x}^\iota \\ \text{s.t. } H_{\bar{x},t}^\iota \bar{x}^\iota \leq h_{\bar{x},t}^\iota \\ \quad H_y^\iota \bar{x}^\iota \leq \bar{p}_y^\iota - \bar{P}_y^\iota \tilde{f}_y^\iota \end{bmatrix} \leq \bar{p}_x^\iota - \bar{P}_x^\iota \tilde{f}_x^\iota - \bar{P}_x^\iota \Gamma_{xu}^\iota u_T. \tag{26}$$

Thus, we can convert the above semi-infinite constraint into a tractable formulation for each time instant $t \in \mathbb{Z}_{T-1}^0$ by using tools from robust optimization [18], [19] to obtain its robust counterpart, as shown in problem (P_{PPDID}).

Moreover, concatenating the separation condition of the inner problem (P_{Inner}) with time in each model pair $\iota \in \mathbb{Z}_I^+$ at time instant $t \in \mathbb{Z}_{T-1}^0$, the constraint (23f) becomes

$$\Lambda^\iota \bar{x}^\iota \leq \delta^\iota - \bar{E}^\iota \tilde{f}^\iota - (\bar{E}^\iota \Gamma_u^\iota + F_u^\iota) u_T, \tag{27}$$

where Λ^ι, $\bar{E}^\iota$, $\tilde{f}^\iota$, Γ_u^ι and F_u^ι are matrices related to the separability condition that will be defined in the appendix. In addition, we can concatenate the inequalities associated with $\bar{x}^\iota$, according to whether they are explicitly dependent on u_T or not:

$$R^\iota \bar{x}^\iota \leq \begin{bmatrix} \mathbf{0} \\ \mathbb{1} \end{bmatrix} \delta^\iota + r^\iota - S^\iota u_T, \quad \text{[Explicitly dependent on } u_T] \tag{28}$$

$$H_{\bar{x},t}^\iota \bar{x}^\iota \leq h_{\bar{x},t}^\iota, \qquad \text{[Implicitly dependent on } u_T] \tag{29}$$

where we define

$$R^\iota = \begin{bmatrix} H_y^\iota \\ \Lambda^\iota \end{bmatrix}, r^\iota = \begin{bmatrix} \bar{p}_y^\iota - \bar{P}_y^\iota \tilde{f}_y^\iota \\ -\bar{E}^\iota \tilde{f}^\iota \end{bmatrix}, S^\iota = \begin{bmatrix} \bar{P}_y^\iota \Gamma_{yu}^\iota \\ \bar{E}^\iota \Gamma_u^\iota + F_u^\iota \end{bmatrix}.$$

Thus, the inner problem (P_{Inner}) for each $\iota \in I$ in Lemma 1 can be written in the concatenated form:

$$\delta^{\iota *}(u_T) = \min_{\delta^\iota, \bar{x}^\iota} \delta^\iota \qquad (P_{Inner})$$

$$\text{s.t.} \quad R^\iota \bar{x}^\iota \leq \begin{bmatrix} \mathbf{0} \\ \mathbb{1} \end{bmatrix} \delta^\iota + r^\iota - S^\iota u_T, \tag{30a}$$

$$H_{\bar{x},t}^\iota \bar{x}^\iota \leq h_{\bar{x},t}^\iota, \tag{30b}$$

$$\forall k \in \mathbb{Z}_{t-1}^0 : \ u(k) = u_T^{*,t-1}(k). \tag{30c}$$

Then, by applying KKT conditions to (P_{Inner}) and rewriting the complementary slackness constraints in the KKT conditions as SOS-1 constraints, we obtain the constraints in Problem (P_{PPDID}). Thus, for each time instant $t \in \mathbb{Z}_{T-1}^0$, we have converted the bilevel problem defined in Problem 1 into the single level MILP problem (P_{PPDID}). □

Note that in comparison to a similar formulation without parameters in [9], the 'responsibility' of the controlled input (7) is enforced in the outer problem, as is required by the problem we are addressing, and as a result, it trivially holds in the inner problem and needs not be explicitly included there. Moreover, it is worth reiterating that any pair of trajectories that share the same node at time instant t on the partition tree must have the same optimal input subsequence up to t.

V. Simulation Examples

In this section, we apply our proposed approach to the highway lane changing scenario (modified from [9]), where the underlying goal is to detect the intention of other road participants so as to improve driving safety and performance.

A. Parametric Model for a Lane Changing Scenario

Similar to the example in [9], we assume that the other vehicle always drives in the center of its lane and hence has no motion in the lateral direction. We also assume that the lane width is $3.2m$. Under these assumptions, the discrete-time equations of motion for the ego and other vehicles are:

$$\begin{aligned}
x_e(k+1) &= x_e(k) + v_{x,e}(k)\delta t,\\
v_{x,e}(k+1) &= (1 - K\delta t)v_{x,e}(k) + u_{x,e}(k)\delta t\\
&\quad + w_{x,e}(k)\delta t + K v_{x,e}^{des}(k)\delta t,\\
y_e(k+1) &= y_e(k) + v_{y,e}(k)\delta t + w_{y,e}(k)\delta t,\\
x_o(k+1) &= x_o(k) + v_{x,o}(k)\delta t,\\
v_{x,o}(k+1) &= v_{x,o}(k) + d_i(k)\delta t + w_{x,o}(k)\delta t,
\end{aligned}$$

where, respectively, x_e and y_e, and $v_{x,e}$ and $v_{y,e}$ are the ego car's longitudinal and lateral positions in m, and the ego car's longitudinal and lateral velocities in $\frac{m}{s}$. x_o and $v_{x,o}$ are the other car's longitudinal position in m and longitudinal velocity in $\frac{m}{s}$, while $u_{x,e}$ and d_i are ego car and other car's acceleration inputs in $\frac{m}{s^2}$, K is a constant feedback control gain that forces the ego car to follow its desired longitudinal velocity reference $v_{x,e}^{des}(k)$ (model-independent), $w_{x,e}$, $w_{x,e}$ and $w_{x,e}$ are process noise signals in $\frac{m}{s^2}$ and δt is the sampling time in s. As discussed in Example 1, we consider the ego car's time-varying reference $v_{x,e}^{des}(k)$ as a parametric variable in our vehicle models. In our simulations, we assume that the controlled inputs for model discrimination are $u_{x,e}(k) \in \mathcal{U}_x \equiv [-7.85, 3.97]\frac{m}{s^2}$

and $v_{y,e}(k) \in \mathcal{U}_y \equiv [-0.35, 0]\frac{m}{s}$ (where y is in the direction away from the other lane), as well as a control gain of $K = 1$, a desired velocity range of $v_{x,e}^{des} \in [29, 33]\frac{m}{s}$, and a sampling time of $\delta t = 0.3s$. In addition, we assume that we have a noisy observation $z(k) = v_{x,o}(k) + v(k)$.

We consider three driver intention models $i \in \{\text{I , C, M}\}$: **Inattentive Driver** (i = I), who fails to notice the ego vehicle and tries to maintain his driving speed, thus proceeding with an acceleration input which lies in a small range $d_I(k) \in \mathcal{D}_I \equiv 10\% \cdot \mathcal{U}$:

$$A_I = \begin{bmatrix} 1 & \delta t & 0 & 0 & 0 \\ 0 & 1-K\delta t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \delta t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B_I = \begin{bmatrix} 0 & 0 & 0 \\ \delta t & 0 & 0 \\ 0 & \delta t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta t \end{bmatrix},$$

$$B_{v_{x,e,I}^{des}} = \begin{bmatrix} 0 & K\delta t & 0 & 0 & 0 \end{bmatrix}^\intercal, f_I = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\intercal,$$

$$B_{w,I} = B_I, C_I = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}, D_I = 0, D_{v,I} = 1.$$

Cautious Driver (i = C), who tends to yield the lane to the ego car with the input equal to $-K_{d,C}(v_{x,e}(k) - v_{x,o}(k)) - L_{p,C}(\bar{y} - y_e(k)) + L_{d,C}v_{y,e}(k) + d_C(k)$, where $K_{d,C} = 0.9$, $L_{p,C} = 2.5$ and $L_{d,C} = 8.9$ are PD controller parameters, $\bar{y} = 2$ and the input uncertainty is $d_C(k) \in \mathcal{D}_C \equiv 5\% \cdot \mathcal{U}$:

$$A_C = \begin{bmatrix} 1 & \delta t & 0 & 0 & 0 \\ 0 & 1-K\delta t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \delta t \\ 0 & -K_{d,C}\delta t & L_{p,C}\delta t & 0 & 1+K_{d,C}\delta t \end{bmatrix},$$

$$B_C = \begin{bmatrix} 0 & 0 & 0 \\ \delta t & 0 & 0 \\ 0 & \delta t & 0 \\ 0 & 0 & 0 \\ 0 & L_{d,C}\delta t & \delta t \end{bmatrix}, f_C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -L_{p,C}\bar{y}\delta t \end{bmatrix},$$

$$B_{v_{x,e,C}^{des}} = B_{v_{x,e,I}^{des}}, B_{w,C} = B_{w,I}, \; C_C = C_I,$$

$$D_C = D_I, D_{v,C} = D_{v,I}.$$

Malicious Driver (i = M), who does not want to yield the lane and attempts to cause a collision with input equal to $K_{d,M}(v_{x,e}(k) - v_{x,o}(k)) + L_{p,M}(\bar{y} - y_e(k)) - L_{d,M}v_{y,e}(k) + d_M(k)$, if provoked, where $K_{d,M} = 1.1$, $L_{p,M} = 2.0$ and $L_{d,M} = 8.7$ are PD controller parameters, $\bar{y} = 2$ and the input uncertainty satisfies $d_M(k) \in \mathcal{D}_M \equiv 5\% \cdot \mathcal{U}$:

$$A_M = \begin{bmatrix} 1 & \delta t & 0 & 0 & 0 \\ 0 & 1-K\delta t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \delta t \\ 0 & K_{d,M}\delta t & -L_{p,M}\delta t & 0 & 1-K_{d,M}\delta t \end{bmatrix},$$

$$B_M = \begin{bmatrix} 0 & 0 & 0 \\ \delta t & 0 & 0 \\ 0 & \delta t & 0 \\ 0 & 0 & 0 \\ 0 & -L_{d,M}\delta t & \delta t \end{bmatrix}, f_M = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ L_{p,M}\bar{y}\delta t \end{bmatrix},$$

$$B_{v_{x,e,M}^{des}} = B_{v_{x,e,I}^{des}}, B_{w,M} = B_{w,I}, C_M = C_I,$$

$$D_M = D_I, D_{v,M} = D_{v,I}.$$

Without loss of generality, we assume that the initial position of the ego car is 0, and the initial position of the other car is constrained by their initial relative distance. The initial velocities of the cars are also constrained to match typical speed limits of the highway. Further, we assume that at the beginning, both cars are close to the center of the lanes. In this case, the initial conditions are as follows:

$$\begin{aligned} v_{x,e}(0) &\in [30, 32]\tfrac{m}{s}, \quad y_e(0) \in [1.1, 1.8]m, \\ v_{x,o}(0) &\in [30, 32]\tfrac{m}{s}, \quad x_o(0) \in [7, 12]m. \end{aligned} \tag{31}$$

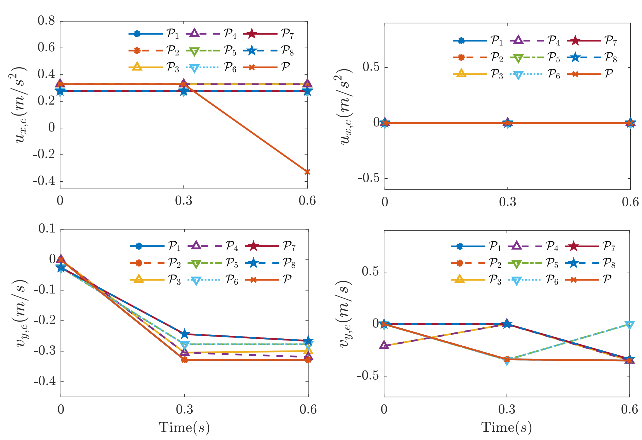
Moreover, the velocity of the ego vehicle is constrained between $[27, 35]\frac{m}{s}$ at all times to obey the speed limit of a highway and the lateral position of the ego vehicle is constrained between $[0.5, 2]m$. Process and measurement noise signals are also limited to the range of $[-0.01, 0.01]$ and the separability threshold is set to $\epsilon = 0.1\frac{m}{s}$.

B. Simulation Results and Discussions

The operating region of $v_{x,e}^{des}(k)$ over the entire time horizon $T = 3$ is $\mathcal{P} = [29, 33]$ and for convenience, we choose to partition the operating region into two subregions that are identical for each time instant, i.e., $\mathcal{P}_1(k) = [29, 31]$ and $\mathcal{P}_2(k) = [31, 33]$ for $k = 0, 1, 2$. In total, we have 2^T different trajectories, as depicted in the partition tree over the entire horizon in Fig. 1.

For our simulations, we consider two optimization costs: (i) $\|u_T\|_1$ that enforces sparsity (leads to minimal number of non-zero inputs) and (ii) $\|u_T\|_\infty$ that ensures comfort (with small maximum input amplitudes). For each trajectory, an optimal separating input is computed *offline* that guarantees the velocities of the other vehicles under each intention are different by at least $\epsilon = 0.1$.

As observed in Fig. 1, by using the partition-based method, the cost at each node of the partition tree is reduced monotonically from its parent node. In particular, this means that our proposed approach improves on the performance of the approach that does not take the revealed information into account, shown at the top of the partition tree. When a certain node does not improve on the cost from its parent (e.g., at $t =$



(a) Input u_T with $J = \|u_T\|_\infty$ (b) Input u_T with $J = \|u_T\|_1$

Fig. 2: Trajectories of the controlled inputs when the operating range of the revealed parameter $p_m(k) = v_{x,e}^{des}(k)$ is partitioned into two subregions at each time. In figure legend, trajectories $\mathcal{P}_1$-$\mathcal{P}_8$ indicate different combinations of the partitions over the entire horizon and are defined as $\mathcal{P}_1 = \{\mathcal{P}_1(0), \mathcal{P}_1(1), \mathcal{P}_1(2)\}$, $\mathcal{P}_2 = \{\mathcal{P}_1(0), \mathcal{P}_1(1), \mathcal{P}_2(2)\}$, $\mathcal{P}_3 = \{\mathcal{P}_1(0), \mathcal{P}_2(1), \mathcal{P}_1(2)\}$, $\mathcal{P}_4 = \{\mathcal{P}_1(0), \mathcal{P}_2(1), \mathcal{P}_2(2)\}$, $\mathcal{P}_5 = \{\mathcal{P}_2(0), \mathcal{P}_1(1), \mathcal{P}_1(2)\}$, $\mathcal{P}_6 = \{\mathcal{P}_2(0), \mathcal{P}_1(1), \mathcal{P}_2(2)\}$, $\mathcal{P}_7 = \{\mathcal{P}_2(0), \mathcal{P}_2(1), \mathcal{P}_1(2)\}$, $\mathcal{P}_8 = \{\mathcal{P}_2(0), \mathcal{P}_2(1), \mathcal{P}_2(2)\}$. In addition, $\mathcal{P}$ corresponds to the entire operating region.

1 or $t = 2$), it means the additional revealed parameter does not change the optimal input that is needed for separation. This can also be observed in Fig. 2, which shows the optimal input sequence associated with each trajectory.

VI. Conclusion

In this paper, a partition-based parametric active model discrimination approach was proposed for computing a sequence of optimal inputs that guarantee optimal separation/discrimination among a set of discrete-time affine time-invariant models with uncontrolled inputs, model-independent parameters and noise signals over a fixed time horizon, where the parameters are revealed in real-time. The approach allows us to take advantage of real time information (i.e., the revealed parameters) to improve the model discrimination performance. Moreover, we move the computation of the optimal separating input offline by considering the revealed parameters as parametric variables and introducing partitions of its operating region. Leveraging tools from robust optimization and applying double negation and KKT conditions, we formulate the offline input design problem as a sequence of tractable MILP problems. We demonstrated our proposed approach on an example of intention estimation in a lane changing scenario, showing that the proposed approach outperforms the approach that does not use the real-time revealed information about the parameters.

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Appendix

In this appendix, we provide definitions of matrices and vectors that were previously omitted to improve readability.

A. Time-Concatenated Matrices and Vectors in Section II-B:

$$\overline{A}_{i,T} = \begin{bmatrix} A_i \\ A_i^2 \\ \vdots \\ A_i^T \end{bmatrix}, \quad \Theta_{i,T} = \begin{bmatrix} \mathbb{I} & 0 & \cdots & 0 \\ A_i & \mathbb{I} & \cdots & 0 \\ \vdots & & \ddots & \\ A_i^{T-1} & A_i^{T-2} & \cdots & \mathbb{I} \end{bmatrix},$$

$$\overline{f}_{i,T} = \operatorname*{vec}_T\{f_i\},\ \tilde{f}_{i,T} = \Theta_{i,T}\overline{f}_{i,T},\ \tilde{g}_{i,T} = \operatorname*{vec}_T\{g_i\},$$

$$E_i = \operatorname*{diag}_T\{C_i\},\ F_{u,i} = \operatorname*{diag}_T\{D_{u,i}\},\ F_{d,i} = \operatorname*{diag}_T\{D_{d,i}\},$$

$$F_{v,i} = \operatorname*{diag}_T\{D_{v,i}\},\ A_{x,i} = \begin{bmatrix} A_{xx,i} \\ A_{yx,i} \end{bmatrix},\ A_{y,i} = \begin{bmatrix} A_{xy,i} \\ A_{yy,i} \end{bmatrix}.$$

For $\dagger = \{x, y\}$ and $\star = \{u, d, p, w\}$:

$$B_{\star,i} = \begin{bmatrix} B_{x\star,i} \\ B_{y\star,i} \end{bmatrix},\ B_{\dagger\star,\mathrm{d},i,T} = \operatorname*{diag}_T\{B_{\dagger\star,i}\},$$

$$\Gamma_{\star,i,T} = \begin{bmatrix} B_{\star,i} & 0 & \cdots & 0 \\ A_i B_{\star,i} & B_{\star,i} & \cdots & 0 \\ \vdots & & \ddots & \\ A_i^{T-1}B_{\star,i} & A_i^{T-2}B_{\star,i} & \cdots & B_{\star,i} \end{bmatrix},$$

$$A_{\dagger,\mathrm{d},i,T} = \operatorname*{diag}_T\{\begin{bmatrix} A_{\dagger x,i} & A_{\dagger y,i} \end{bmatrix}\},$$

$$M_{\dagger,i,T} = A_{\dagger,\mathrm{d},i,T}\begin{bmatrix} \mathbb{I} \\ \overline{A}_{i,T-1} \end{bmatrix},\ \overline{f}_{\dagger,i,T} = \operatorname*{vec}_T\{f_{\dagger,i}\},$$

$$\tilde{f}_{\dagger,i,T} = A_{\dagger,\mathrm{d},i,T}\begin{bmatrix} 0 \\ \Theta_{i,T-1} \end{bmatrix}\overline{f}_{i,T-1} + \overline{f}_{\dagger,i,T},$$

$$\Gamma_{\dagger\star,i,T} = A_{\dagger,\mathrm{d},i,T}\begin{bmatrix} 0 & 0 \\ \Gamma_{\star,i,T-1} & 0 \end{bmatrix} + B_{\dagger\star,\mathrm{d},i,T}.$$

B. Matrices and Vectors in Theorem 1:

$$\overline{A}^\iota = \operatorname*{diag}_{i,j}\{\overline{A}_{i,T}\},\ \overline{C}^\iota = \operatorname*{diag}_{i,j}\{E_i\},\ \tilde{g}^\iota = \operatorname*{vec}_{i,j}\{\tilde{g}_{i,T}\},$$

$$\Gamma_u^\iota = \operatorname*{vec}_{i,j}\{\Gamma_{u,i,T}\},\ \Gamma_d^\iota = \operatorname*{diag}_{i,j}\{\Gamma_{d,i,T}\},\ \Gamma_w^\iota = \operatorname*{diag}_{i,j}\{\Gamma_{w,i,T}\},$$

$$\Gamma_p^\iota = \operatorname*{diag}_{i,j}\{\Gamma_{p,i,T}\},\ \tilde{f}^\iota = \operatorname*{vec}_{i,j}\{\tilde{f}_{i,T}\},\ \overline{D}_u^\iota = \operatorname*{vec}_{i,j}\{F_{u,i}\},$$

$$\overline{D}_d^\iota = \operatorname*{diag}_{i,j}\{F_{d,i}\},\ \overline{D}_v^\iota = \operatorname*{diag}_{i,j}\{F_{v,i}\},\ \overline{E}^\iota = \begin{bmatrix} E_i & -E_j \\ -E_i & E_j \end{bmatrix},$$

$$\Lambda^\iota = \overline{E}^\iota\begin{bmatrix} \overline{A}^\iota & \Gamma_d^\iota & \Gamma_w^\iota & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \overline{F}_d^\iota & \mathbf{0} & \overline{F}_v^\iota \end{bmatrix}.$$

For $\dagger = \{x, y\}$:

$$\Gamma_{\dagger u}^\iota = \operatorname*{vec}_{i,j}\{\Gamma_{\dagger u,i,T}\}, \Gamma_{\dagger d}^\iota = \operatorname*{diag}_{i,j}\{\Gamma_{\dagger d,i,T}\}, \Gamma_{\dagger p}^\iota = \operatorname*{vec}_{i,j}\{\Gamma_{\dagger p,i,T}\},$$

$$\Gamma_{\dagger w}^\iota = \operatorname*{diag}_{i,j}\{\Gamma_{\dagger w,i,T}\}, M_\dagger^\iota = \operatorname*{diag}_{i,j}\{M_{\dagger,i,T}\}, \tilde{f}_\dagger^\iota = \operatorname*{vec}_{i,j}\{\tilde{f}_{\dagger,i,T}\}.$$

For $* = \{d, v\}$:

$$\overline{F}_*^\iota = \begin{bmatrix} F_{*,i} & -F_{*,j} \\ -F_{*,i} & F_{*,j} \end{bmatrix}, \overline{F}_u^\iota = \begin{bmatrix} F_{u,i} - F_{u,j} \\ F_{u,j} - F_{u,i} \end{bmatrix}, \overline{g}^\iota = \begin{bmatrix} \tilde{g}_i - \tilde{g}_j \\ -\tilde{g}_i + \tilde{g}_j \end{bmatrix}.$$