

Robust Optimization-Based Affine Abstractions for Uncertain Affine Dynamics

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Abstract—This paper considers affine abstractions for over-approximating uncertain affine discrete-time systems, where the system uncertainties are represented by interval matrices, by a pair of affine functions in the sense of inclusion of all possible trajectories over the entire domain. The affine abstraction problem is a robust optimization problem with nonlinear uncertainties. To make this problem practically solvable, we convert the nonlinear uncertainties into linear uncertainties by exploiting the fact that the system uncertainties are hyperrectangles and thus, we only need to consider the vertices of the hyperrectangles instead of the entire uncertainty sets. Hence, affine abstraction can be solved efficiently by computing its corresponding robust counterpart to obtain a linear programming problem. Finally, we demonstrate the effectiveness of the proposed approach for abstracting uncertain driver intention models in an intersection crossing scenario.

I. INTRODUCTION

Nowadays, cyber-physical systems such as smart grids, autonomous vehicles and smart building are becoming increasingly complex, integrated and interconnected. One of the difficulties in designing cyber-physical systems is their complex dynamics, which is almost always nonlinear, uncertain or hybrid. To deal with this, abstraction approaches for cyber-physical systems to approximate the original complex dynamics with simpler dynamics have gained increased popularity over the last few years [1]–[3]. The abstraction approaches compute a simple but conservative approximation that can be used to represent the original dynamics and allow one to apply the well-developed controller or observer design methods, especially in the cases where reachability and safety specifications for controller synthesis or guarantees for estimator design are needed.

Literature Review. The key idea of abstraction is to find a new simpler system that shares most properties of interest with the original system dynamics [4]. The abstraction has been studied for various classes of systems, for example, linear systems [5], nonlinear systems [6]–[8], incrementally stable switched systems [9], and discrete-time hybrid systems [10]. In general, the abstraction process partitions the state space of the original complex system into finite regions, and a simple abstract model, which may be non-holonomic chained-form [11], piecewise-affine [8] and multi-affine [12], is assigned to over-approximate the original system in the

sense of the inclusion of all possible trajectories in each region. Since the dynamics of the abstracted system changes when the system state moves among different regions, the abstraction could also be considered as a hybridization process [13]. In [8], the original nonlinear dynamics is conservatively approximated by a linear affine system with bounded disturbances on each simplex of the triangulation of the whole state space, where the disturbances account for approximation errors and ensure the conservativeness of the approximation. In [14], [15], the dynamic on-the-fly abstraction method is proposed, where the domain construction and the abstraction process are only carried out on states that are reachable. In [10], a pair of piecewise affine functions is computed to over-approximate a nonlinear Lipschitz continuous function over a bounded region such that the synthesized controllers for the abstracted systems are guaranteed to be controllers for the original systems. In [16], an optimization-based approach is used to find linear uncertain affine abstractions for nonlinear models without partitioning the state space, which preserve all the system characteristics such that the any model discrimination guarantees for the uncertain affine abstraction also hold for the original nonlinear systems. In [18], the problem of piecewise affine abstraction of nonlinear functions with different degrees of smoothness is solved by using a mesh-based method. However, none of the above-mentioned abstraction approaches is applicable for over-approximating uncertain affine dynamics.

Contributions. In this paper, we propose a robust optimization based affine abstraction approach to conservatively approximate uncertain affine discrete-time systems in the sense of the inclusion of all possible trajectories by a pair of affine functions over the whole state space. We assume that all system matrices of the affine discrete-time system are uncertain, where the uncertainty is represented by interval matrices/vectors and equivalently by hyperrectangles. To over-approximate the uncertain behavior over the entire domain, two affine functions are constructed as upper and lower bounds to the original dynamics instead of only having one interval-valued affine function with a bounded error set, as is done in the hybridization approaches in [8], [17]. At first glance, the abstraction problem results in a robust optimization program with nonlinear uncertainties, which is not practically solvable. However, since the uncertainties about the system matrices are in the form of hyperrectangles, we propose to convert the nonlinear uncertainties into linear uncertainties by only using the vertices of the hyperrectangles. Then, we can leverage tools from robust optimization to solve the abstraction problem in a computationally

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tractable manner. Comparing with our recent optimization-based abstraction method for nonlinear system in [16], this method can achieve affine abstraction by solving a linear programming (LP) optimization, and hence the abstraction efficiency is improved and the optimality gap is eliminated.

II. PRELIMINARIES

A. Notation

For a vector $v \in \mathbb{R}^n$, $\|v\|_1$ denotes the vector 1-norm. An interval matrix M^I is defined as a set of matrices of the form $M^I = [M_l, M_u] = \{M \in \mathbb{R}^{n \times m} : M_l \leq M \leq M_u\}$, where M_l and M_u are $n \times m$ matrices, and the inequality is to be understood componentwise. If A is a interval matrix with elements $[a_{l,ij}, a_{u,ij}]$ and B is a matrix with real elements b_{ij} such that $b_{ij} \in [a_{l,ij}, a_{u,ij}]$ for all i and j , then we write $B \in A$. We denote $[n]$ as the initial segment $1, \dots, n$ of the natural numbers, $|\mathcal{X}|$ as the cardinality of a set \mathcal{X} , and I_n as a $n \times n$ identity matrix.

B. Modeling Framework

Consider an uncertain affine discrete-time system:

$$x(k+1) = Ax(k) + B_u u(k) + B_w w(k) + B_f f, \quad (1)$$

where $x(k) \in \mathcal{X} \subset \mathbb{R}^n$ is the system state, $u(k) \in \mathcal{U} \subset \mathbb{R}^{m_u}$ is the control input, and $w(k) \in \mathcal{W} \subset \mathbb{R}^{m_w}$ is the process noise at the current time instant k , the vector $f \in \mathcal{F} \subset \mathbb{R}^{m_f}$ is an unknown additive constant. Throughout the paper, we assume that the domain \mathcal{X} , \mathcal{U} , \mathcal{W} and \mathcal{F} are polyhedral sets:

$$\mathcal{X} = \{x \in \mathbb{R}^n : Q_x x \leq q_x\}, \quad (2a)$$

$$\mathcal{U} = \{u \in \mathbb{R}^{m_u} : Q_u u \leq q_u\}, \quad (2b)$$

$$\mathcal{W} = \{w \in \mathbb{R}^{m_w} : Q_w w \leq q_w\}, \quad (2c)$$

$$\mathcal{F} = \{f \in \mathbb{R}^{m_f} : Q_f f \leq q_f\}, \quad (2d)$$

where the matrices $Q_x \in \mathbb{R}^{k_x \times n}$, $Q_u \in \mathbb{R}^{k_u \times m_u}$, $Q_w \in \mathbb{R}^{k_w \times m_w}$, $Q_f \in \mathbb{R}^{k_f \times m_f}$, and the vectors $q_x \in \mathbb{R}^{k_x}$, $q_u \in \mathbb{R}^{k_u}$, $q_w \in \mathbb{R}^{k_w}$ and $q_f \in \mathbb{R}^{k_f}$ are constant, and imposed by the desired domain of operation/observation or to describe physical constraints. Due to measurement errors or component tolerances, the state matrix $A \in \mathbb{R}^{n \times n}$, input matrix $B \in \mathbb{R}^{n \times m}$, noise matrix $B_w \in \mathbb{R}^{n \times m_w}$, and fault matrix $B_f \in \mathbb{R}^{n \times m_f}$ are uncertain and known to the extent of

$$A \in A^I = [A_l, A_u], B \in B^I = [B_l, B_u], \quad (3a)$$

$$B_w \in B_w^I = [B_{w,l}, B_{w,u}], B_f \in B_f^I = [B_{f,l}, B_{f,u}], \quad (3b)$$

where the interval matrices or vectors A^I , B^I , B_w^I and B_f^I define the ranges of the uncertainties.

Consequently, for compactness, the uncertain linear discrete-time system (1) is further rewritten as

$$x(k+1) = Gh(k), \quad (4)$$

with an augmented state $h(k) = [x^T(k) \ u^T(k) \ w^T(k) \ f^T]^T \in \mathbb{R}^\xi$ and an augmented uncertain system matrix $G = [A \ B \ B_w \ B_f] \in \mathbb{R}^{n \times \xi}$ with $\xi = n + m + m_w + m_f$. In view of (2), it is clear that $h(k) \in \mathcal{H} \subset \mathbb{R}^\xi$, which is also a polyhedral set given as

$$\mathcal{H} = \{h \in \mathbb{R}^\xi : Qh \leq q\}, \quad (5)$$

where $Q = \text{diag}(Q_x, Q_u, Q_w, Q_f) \in \mathbb{R}^{k \times \xi}$, $q = [q_x^T \ q_u^T \ q_w^T \ q_f^T]^T \in \mathbb{R}^k$ and $k = k_x + k_u + k_w + k_f$. Moreover, considering the system uncertainties defined by interval matrices in (3), the augmented system matrix G satisfies

$$G \in G^I = [G_l, G_u], \quad (6)$$

where $G_l = [A_l \ B_l \ B_{w,l} \ B_{f,l}] \in \mathbb{R}^{n \times \xi}$ and $G_u = [A_u \ B_u \ B_{w,u} \ B_{f,u}] \in \mathbb{R}^{n \times \xi}$.

C. Problem Statement

In this paper, we aim to over-approximate/abstract the uncertain affine discrete-time dynamics by a pair of affine hyperplanes (\underline{f}, \bar{f}) such that for all $G \in G^I$ and $h(k) \in \mathcal{H}$, we have that $\underline{f} \leq Gh(k) \leq \bar{f}$ (i.e., $Gh(k)$ is lower and upper bounded by \underline{f} and \bar{f}). As a result, the uncertain affine system defined in (1) lies between the lower and upper affine hyperplanes, which are defined as

$$\underline{f}(h(k)) = \underline{G}h(k) + \underline{b}, \bar{f}(h(k)) = \bar{G}h(k) + \bar{b}, \quad (7)$$

where the matrices \underline{G} and \bar{G} , and the vectors \underline{b} and \bar{b} are constant and of appropriate dimensions. We say that an affine plane pair (\underline{f}, \bar{f}) over-approximates/abstracts the uncertain affine dynamics if $\underline{f}(h(k)) \leq Gh(k) \leq \bar{f}(h(k))$, $\forall G \in G^I$ and $\forall h(k) \in \mathcal{H}$. The affine hyperplanes pair (\underline{f}, \bar{f}) is then the affine abstraction of the uncertain affine dynamics.

Definition 1 (Approximation Error): If a pair of affine hyperplanes (\underline{f}, \bar{f}) over-approximates an uncertain affine discrete-time dynamics defined in (4) over the system constraints $h(k) \in \mathcal{H}$ and uncertainties $G \in G^I$, then the approximation error of the abstraction with respect to the uncertain affine dynamics is defined as

$$e(\underline{f}, \bar{f}) = \max_{h(k) \in \mathcal{H}} \|\bar{f}(h(k)) - \underline{f}(h(k))\|_1. \quad (8)$$

Problem 1 (Affine Abstraction): For an uncertain affine discrete-time dynamics given in (4) with the polyhedral domain $h(k) \in \mathcal{H}$ and the uncertain system matrices $G \in G^I$, the affine abstraction is to find an affine hyperplane pair (\underline{f}, \bar{f}) (i.e., the lower and upper hyperplanes defined in (7)) to over-approximate/abstract the given uncertain affine dynamics with the minimum approximation error. Thus, the affine abstraction problem for uncertain linear discrete-time systems is equivalent to a robust optimization problem:

$$\begin{aligned} \min_{\theta, \underline{G}, \underline{b}, \bar{b}} \quad & \theta \\ \text{s.t.} \quad & \bar{G}h(k) + \bar{b} \geq Gh(k), \\ & \forall G \in G^I, \forall h(k) \in \mathcal{H}, \end{aligned} \quad (9a)$$

$$\begin{aligned} & \underline{G}h(k) + \underline{b} \leq Gh(k), \\ & \forall G \in G^I, \forall h(k) \in \mathcal{H}, \end{aligned} \quad (9b)$$

$$\max_{h(k) \in \mathcal{H}} \|\bar{f}(h(k)) - \underline{f}(h(k))\|_1 \leq \theta, \quad (9c)$$

where θ is the least upper bound on the approximation error.

III. AFFINE ABSTRACTION

The affine abstraction defined in Problem 1 is formulated as a robust optimization program. Since there are nonlinear uncertainties in (9a)-(9b), i.e., multiplication of two uncertain

sets, and the semi-infinite constraints as illustrated in (9a)-(9c), the formulation in Problem 1 is not practically solvable. To cope with this, we leverage the form of the uncertainties associated with the system matrices and tools from robust optimization [19], [20] to convert this problem into an LP, which can be readily and efficiently solved.

A. Equivalent LP via Robustification

Based on the definition of an interval matrix, the uncertainty of each row of the system matrix G can be equivalently written as a ξ -dimensional hyperrectangle that is defined as

$$\mathcal{G}_i = [G_{l,i1}, G_{u,i1}] \times \dots \times [G_{l,i\xi}, G_{u,i\xi}], \quad \forall i \in [n]. \quad (10)$$

Thus, we have $(G)_i^\top \in \mathcal{G}_i$, where $(G)_i \in \mathbb{R}^{1 \times \xi}$ represents the i th row of the matrix G . Consequently, the Problem 1 with row-wise uncertainty can be rewritten as

$$\begin{aligned} \min_{\theta, \bar{G}, \underline{G}, \bar{b}, \underline{b}} \quad & \sum_{i=1}^n \theta_i \\ \text{s.t.} \quad & (\bar{G})_i h(k) + \bar{b}_i \geq (G)_i h(k), \\ & \forall (G)_i^\top \in \mathcal{G}_i, \forall h(k) \in \mathcal{H}, \forall i \in [n], \end{aligned} \quad (11a)$$

$$\begin{aligned} & (\underline{G})_i h(k) + \underline{b}_i \leq (G)_i h(k), \\ & \forall (G)_i^\top \in \mathcal{G}_i, \forall h(k) \in \mathcal{H}, \forall i \in [n], \end{aligned} \quad (11b)$$

$$\begin{aligned} & \bar{f}_i(h(k)) - \underline{f}_i(h(k)) \leq \theta_i, \\ & \forall h(k) \in \mathcal{H}, \forall i \in [n], \end{aligned} \quad (11c)$$

where θ_i is the approximation error of the i th row. Moreover, we derive the following useful lemma:

Lemma 1: Let the vertex set of the ξ -dimensional hyperrectangle \mathcal{G}_i be denoted as $\mathcal{V}_i^g = \{v_{i,1}^g, \dots, v_{i,\varrho}^g\}$ with $\varrho = |\mathcal{V}_i^g| \leq 2^\xi$, where $\varrho = 2^\xi$ holds when G_i is unstructured, i.e., all elements of G_i are independent. The constraints

$$h^\top(k)(\bar{G})_i^\top + \bar{b}_i \geq h^\top(k)v_{i,j}^g, \quad \forall j \in [\varrho], \quad (12)$$

$$h^\top(k)(\underline{G})_i^\top + \underline{b}_i \leq h^\top(k)v_{i,j}^g, \quad \forall j \in [\varrho], \quad (13)$$

are equivalent to $(\bar{G})_i h(k) + \bar{b}_i \geq (G)_i h(k)$, $\forall (G)_i^\top \in \mathcal{G}_i$ and $(\underline{G})_i h(k) + \underline{b}_i \leq (G)_i h(k)$, $\forall (G)_i^\top \in \mathcal{G}_i$.

Proof: Since \mathcal{G}_i is a ξ -dimensional hyperrectangle with vertex set $\mathcal{V}_i^g = \{v_{i,1}^g, \dots, v_{i,\varrho}^g\}$, any point in $(G)_i^\top \in \mathcal{G}_i$ can be represented as

$$(G)_i^\top = \sum_{j=1}^{\varrho} \alpha_j v_{i,j}^g, \quad (14)$$

where $\alpha_j \geq 0$ and $\sum_{j=1}^{\varrho} \alpha_j = 1$. Multiplying both sides of (12) and (13) by the nonnegative constant α_j , we have

$$\alpha_j h^\top(k)(\bar{G})_i^\top + \alpha_j \bar{b}_i \geq \alpha_j h^\top(k)v_{i,j}^g, \quad \forall j \in [\varrho], \quad (15)$$

$$\alpha_j h^\top(k)(\underline{G})_i^\top + \alpha_j \underline{b}_i \leq \alpha_j h^\top(k)v_{i,j}^g, \quad \forall j \in [\varrho]. \quad (16)$$

Furthermore, adding all of the ϱ inequalities in (15) and (16) respectively yields

$$\sum_{j=1}^{\varrho} \alpha_j h^\top(k)(\bar{G})_i^\top + \sum_{j=1}^{\varrho} \alpha_j \bar{b}_i \geq \sum_{j=1}^{\varrho} \alpha_j h^\top(k)v_{i,j}^g, \quad (17)$$

$$\sum_{j=1}^{\varrho} \alpha_j h^\top(k)(\underline{G})_i^\top + \sum_{j=1}^{\varrho} \alpha_j \underline{b}_i \leq \sum_{j=1}^{\varrho} \alpha_j h^\top(k)v_{i,j}^g. \quad (18)$$

In light of $\sum_{j=1}^{\varrho} \alpha_j = 1$ and (14), the sufficiency can be obtained directly. Conversely, suppose we have $(\bar{G})_i h(k) + \bar{b}_i \geq (G)_i h(k)$, $\forall (G)_i^\top \in \mathcal{G}_i$ and $(\underline{G})_i h(k) + \underline{b}_i \leq (G)_i h(k)$, $\forall (G)_i^\top \in \mathcal{G}_i$. As the uncertain set \mathcal{G}_i contains every point including all its vertices, it is obvious that (12) and (13) hold. This completes the proof. \blacksquare

Theorem 1: The affine abstraction problem defined in Problem 1 is equivalent to the following LP problem:

$$\min_{\theta, \bar{G}, \underline{G}, \bar{b}, \underline{b}, \bar{p}_{i,j}, \underline{p}_{i,j}, \Pi} \quad \sum_{i=1}^n \theta_i \quad (P_{AB})$$

$$\text{s.t.} \quad \bar{p}_{i,j}^\top q \leq \bar{b}_i, \quad \forall i \in [n], \forall j \in [\varrho], \quad (19a)$$

$$Q^\top \bar{p}_{i,j} = v_{i,j}^g - (\bar{G})_i^\top, \quad \forall i \in [n], \forall j \in [\varrho], \quad (19b)$$

$$\bar{p}_{i,j} \geq 0, \quad \forall i \in [n], \forall j \in [\varrho], \quad (19c)$$

$$\underline{p}_{i,j}^\top q \leq -\underline{b}_i, \quad \forall i \in [n], \forall j \in [\varrho], \quad (19d)$$

$$Q^\top \underline{p}_{i,j} = (\underline{G})_i^\top - v_{i,j}^g, \quad \forall i \in [n], \forall j \in [\varrho], \quad (19e)$$

$$\underline{p}_{i,j} \geq 0, \quad \forall i \in [n], \forall j \in [\varrho], \quad (19f)$$

$$\Pi_i^\top q \leq \theta_i + \underline{b}_i - \bar{b}_i, \quad (19g)$$

$$\Pi_i^\top Q = (\bar{G})_i - (\underline{G})_i, \quad (19h)$$

$$\Pi_i \geq 0, \quad (19i)$$

where $\bar{p}_{i,j}$, $\underline{p}_{i,j}$ and Π are dual variables.

Proof: Lemma 1 implies that we can only consider all the vertices of the ξ -dimensional hyperrectangle \mathcal{G}_i instead of every point in it. This simplifies the multiplication of two uncertain variables in constraints (11a) and (11b), so that the nonlinear uncertainty in Problem 1 reduces to a linear uncertainty. As a consequence, the Problem 1 can be further cast as

$$\min_{\theta, \bar{G}, \underline{G}, \bar{b}, \underline{b}} \quad \sum_{i=1}^n \theta_i$$

$$\text{s.t.} \quad h^\top(k)(\bar{G})_i^\top + \bar{b}_i \geq h^\top(k)v_{i,j}^g, \quad \forall h(k) \in \mathcal{H}, \forall i \in [n], \forall j \in [\varrho], \quad (20a)$$

$$h^\top(k)(\underline{G})_i^\top + \underline{b}_i \leq h^\top(k)v_{i,j}^g, \quad \forall h(k) \in \mathcal{H}, \forall i \in [n], \forall j \in [\varrho], \quad (20b)$$

$$\bar{f}_i(h(k)) - \underline{f}_i(h(k)) \leq \theta_i, \quad \forall h(k) \in \mathcal{H}, \forall i \in [n], \quad (20c)$$

where $v_{i,j}^g$, as defined in Lemma 1, denotes the vertex of the ξ -dimensional hyperrectangle \mathcal{G}_i .

Now, the above equivalent abstraction problem is in a standard formulation of the robust optimization, we can use the method presented in [19], [20] to convert the semi-infinite constraints into a tractable formulation. Specifically, for the upper hyperplane constraint in (20a), it can be equivalently written as

$$\left[\begin{array}{l} \max_{h(k) \in \mathbb{R}^\xi} \quad h^\top(k) (v_{i,j}^g - (\bar{G})_i^\top) \\ \text{s.t.} \quad Qh(k) \leq q \\ \forall i \in [n], \forall j \in [\varrho]. \end{array} \right] \leq \bar{b}_i, \quad (21)$$

We proceed by applying LP duality to the inner maximization subproblem, turning it into an inner minimization problem.

Thus, we have

$$\left[\begin{array}{l} \min_{\bar{p}_{i,j} \in \mathbb{R}^k} \bar{p}_{i,j}^\top q \\ \text{s.t.} \quad Q^\top \bar{p}_{i,j} = v_{i,j}^g - (\bar{G})_i^\top \\ \bar{p}_{i,j} \geq 0 \end{array} \right] \leq \bar{b}_i, \quad (22)$$

$\forall i \in [n], \forall j \in [\varrho].$

Similarly, the lower hyperplane constraints in (20b) can also be rewritten as

$$\left[\begin{array}{l} \max_{h(k) \in \mathbb{R}^\xi} h^\top(k) ((\underline{G})_i^\top - v_{i,j}^g) \\ \text{s.t.} \quad Qh(k) \leq q \end{array} \right] \leq -\underline{b}_i, \quad (23)$$

$\forall i \in [n], \forall j \in [\varrho].$

Then, by using linear duality for its inner maximization subproblem, the constraint becomes

$$\left[\begin{array}{l} \min_{p_{i,j} \in \mathbb{R}^k} p_{i,j}^\top q \\ \text{s.t.} \quad Q^\top p_{i,j} = (\underline{G})_i^\top - v_{i,j}^g \\ p_{i,j} \geq 0 \end{array} \right] \leq -\underline{b}_i, \quad (24)$$

$\forall i \in [n], \forall j \in [\varrho].$

Finally, for the constraint describing the upper bound of the approximation error in (20c), we have

$$\left[\begin{array}{l} \max_{h(k) \in \mathbb{R}^\xi} ((\bar{G})_i - (\underline{G})_i) h(k) \\ \text{s.t.} \quad Qh(k) \leq q \end{array} \right] \leq \theta_i + \underline{b}_i - \bar{b}_i, \quad (25)$$

$\forall i \in [n].$

Taking the dual of the above inner maximization leads to

$$\left[\begin{array}{l} \min_{\Pi_i \in \mathbb{R}^k} \Pi_i^\top q \\ \text{s.t.} \quad \Pi_i^\top Q = (\bar{G})_i - (\underline{G})_i \\ \Pi_i \geq 0 \end{array} \right] \leq \theta_i + \underline{b}_i - \bar{b}_i, \quad (26)$$

$\forall i \in [n].$

With these three inner minimization subproblems derived in (22), (25) and (26), we can convert the affine abstraction problem defined in Problem 1 into a tractable problem. Note that dropping the inner minimization operator and regarding the decision variables of the inner minimization as additional variables to the outer minimization would not change the optimal value [19], [20]. Thus, the affine abstraction problem can be equivalently recast to its robust counterpart (RC), which is the single level LP problem (P_{AB}). ■

Since (P_{AB}) is an LP, it can be solved efficiently. As for the computational complexity of the proposed approach (P_{AB}), it is observed that there are $1 + 2n + kn + 2n\xi + kn\varrho$ decision variables with $n(\xi + 1)(2\varrho + 1)$ linear constraints. However, the number of vertices (e.g., ϱ) increases exponentially with respect to the increase of system dimension.

Proposition 1: If the system state $x(k)$, control input $u(k)$, process noise $w(k)$ and the unknown constant vector f are also constrained by closed interval domains $\mathcal{X} = [a_{x,1}, b_{x,1}] \times \dots \times [a_{x,n}, b_{x,n}] \subset \mathbb{R}^n$, $\mathcal{U} = [a_{u,1}, b_{u,1}] \times \dots \times [a_{u,m}, b_{u,m}] \subset \mathbb{R}^m$, $\mathcal{W} = [a_{w,1}, b_{w,1}] \times \dots \times [a_{w,m_w}, b_{w,m_w}] \subset \mathbb{R}^{m_w}$ and $\mathcal{F} = [a_{f,1}, b_{f,1}] \times \dots \times [a_{f,m_f}, b_{f,m_f}] \subset \mathbb{R}^{m_f}$, then the constraints (19g)-(19i) in the affine abstraction problem (P_{AB}) can be replaced by

$$((\bar{G})_i - (\underline{G})_i) v_j^h + \bar{b}_i - \underline{b}_i \leq \theta_i, \quad \forall i \in [n], \forall j \in [2\xi], \quad (27)$$

where $v_j^h \in \mathcal{V}^h = \{v_1^h, \dots, v_{2\xi}^h\}$ is the vertex set of the ξ -dimensional hyperrectangle \mathcal{H}_p defined in the proof.

Proof: Based on the definition of interval matrices, the augmented state $h(k) = [x(k) \ u(k) \ w(k) \ f]^\top \in \mathbb{R}^\xi$ is also constrained by a ξ -dimensional hyperrectangle defined as $\mathcal{H}_p = \mathcal{X} \times \mathcal{U} \times \mathcal{W} \times \mathcal{F}$, which can be equivalently written as polyhedral set $\mathcal{H}_p = \{h \in \mathbb{R}^\xi : Qh \leq q\}$, where

$$Q = [I_\xi \quad -I_\xi]^\top \in \mathbb{R}^{2\xi \times \xi},$$

$$q = [b_x^\top \ b_u^\top \ b_w^\top \ b_f^\top \ -a_x^\top \ -a_u^\top \ -a_w^\top \ -a_f^\top]^\top \in \mathbb{R}^{2\xi}.$$

In this case, for the constraint (20c), consider any one dimension in \mathbb{R}^ξ with the other dimensions arbitrarily fixed. Due to the linear nature of the difference between \bar{f} and \underline{f} , the difference can only be increasing or decreasing as the augmented state moves in one direction. Due to this observation, the maximum difference would be at one of the ends. Since this argument applies to all dimensions, it follows that the maximum difference must be attained at one of the vertices of \mathcal{H}_p . In view of this, we do not have to apply robust optimization to the constraint of the approximation error and only need to minimize the difference among the vertices of the ξ -dimensional hyperrectangle. Therefore, the constraint (20c) can be equivalently replaced by (27). ■

As a result of the additional assumption in Proposition 1, the abstraction problem (P_{AB}) has $1 + 2n + 2n\xi + nk\varrho$ decision variables and $n(2\xi + 2\varrho + 2\xi\varrho)$ linear constraints, which indicates that we have kn less decision variables but $n(2\xi - \xi - 1)$ more linear constraints when compared to the optimization formulation in (19). Note that since ξ is the total number of states, control inputs, noise, and additive faults, $\xi \geq 1$ always holds and hence, $n(2\xi - \xi - 1) \geq 0$.

Remark 1: In most affine systems, the noise matrix B_w and the fault matrix B_f are fixed. Thus, the compact form of the uncertain affine system (1) can be written as

$$x(k+1) = G_1 h_1(k) + G_2 h_2(k),$$

where the augmented uncertain system matrix $G_1 = [A \ B] \in G_1^I \subset \mathbb{R}^{n \times \xi_1}$ and the fixed matrix $G_2 = [B_w \ B_f] \in \mathbb{R}^{n \times \xi_2}$ with $\xi_1 = n + m$ and $\xi_2 = m_w + m_f$, and the corresponding augmented states $h_1(k) = [x^\top(k) \ u^\top(k)]^\top \in \mathcal{H}_1 \subset \mathbb{R}^{\xi_1}$ and $h_2(k) = [w^\top(k) \ f^\top]^\top \in \mathcal{H}_2 \subset \mathbb{R}^{\xi_2}$. Then, the lower and upper hyperplanes for the abstraction are defined as

$$\underline{f} = \underline{G}_1 h_1(k) + \underline{b}_1 + G_2 h_2(k),$$

$$\bar{f} = \bar{G}_1 h_1(k) + \bar{b}_1 + G_2 h_2(k).$$

Then, the affine abstraction is formulated as

$$\begin{array}{ll} \min_{\bar{G}_1, \underline{G}_1, \bar{b}_1, \underline{b}_1, \theta} & \theta \\ \text{s.t.} & \bar{G}_1 h(k) + \bar{b}_1 \geq G_1 h(k), \\ & \forall G_1 \in G_1^I, \forall h_1(k) \in \mathcal{H}_1, \\ & \underline{G}_1 h(k) + \underline{b}_1 \leq G_1 h(k), \\ & \forall G_1 \in G_1^I, \forall h_1(k) \in \mathcal{H}_1, \\ & \max_{h_1(k) \in \mathcal{H}_1} \|\bar{f} - \underline{f}\|_1 \leq \theta. \end{array}$$

Since we only consider the uncertainties on G_1 and $h_1(k)$, the above formulation has a lower dimension and complexity. Following the same procedures in solving the Problem 1, we can also obtain the solution of the above low-dimensional affine abstraction problem (omitted for brevity).

IV. SIMULATION EXAMPLES

In this section, we apply the proposed affine abstraction to over-approximate uncertain intention models of other human-driven vehicles in the scenario of intersection crossing.

A. Vehicle and Intention Models

Consider two vehicles at an intersection, which is the origin of the coordinate system. The discrete-time equations governing the motion of two vehicles are given in [21]:

$$\begin{aligned} x_e(k+1) &= x_e(k) + v_{x,e}(k)\delta t, \\ v_{x,e}(k+1) &= v_{x,e}(k) + u(k)\delta t + w_x(k)\delta t, \\ y_o(k+1) &= y_o(k) + v_{y,o}(k)\delta t, \\ v_{y,o}(k+1) &= v_{y,o}(k) + d_i(k)\delta t + w_y(k)\delta t, \end{aligned}$$

where x_e and $v_{x,e}$ are ego car's position and velocity, y_o and $v_{y,o}$ are other car's position and velocity, w_x and w_y are process noise, and $\delta t = 0.3$ s is the sampling time. u is the acceleration input for the ego car, whereas d_i is the acceleration input of the other car for each intention $i \in \{C, M, I\}$, corresponding to a Cautious, Malicious or Innattentive driver.

As illustrated in [21], we use a PD controller to model driver's intention. However, the control gains in these intention models cannot be exactly obtained due to the complexity of the human's driving behavior. The **Cautious** driver drives carefully and tends to stop at the intersection with an input equal to $d_C \triangleq -K_{p,C}y_o(k) - K_{d,C}v_{y,o}(k) + \tilde{d}_C(k)$, where the uncertain PD controller parameters $K_{p,C} \in [0, 1.8]$ and $K_{d,C} \in [0, 5.5]$ represent characteristics of the cautious driver, and $\tilde{d}_C(k) \in \mathcal{D}_C = [-0.392, 0.198]$ m/s² denotes the unmodeled variations accounting for nondeterministic driving behaviors. The **Malicious** driver drives aggressively and attempts to cause a collision with an input $d_M \triangleq K_{p,M}(x_e(k) - y_o(k)) + K_{d,M}(v_{x,e}(k) - v_{y,o}(k)) + \tilde{d}_M(k)$, where $K_{p,M} \in [0, 2]$ and $K_{d,M} \in [0, 5]$ are PD controller parameters, and $\tilde{d}_M(k) \in \mathcal{D}_M = [-0.392, 0.198]$ m/s². Finally, the **Inattentive** driver is unaware of the ego car and attempts to maintain its speed with an uncontrolled acceleration input $d_I(k) \in \mathcal{D}_I = [-0.784, 0.396]$ m/s².

Substituting the intention models into the dynamics of the other vehicle, the equation of motion governing the other car's velocity under different intentions becomes:

Cautious Driver ($i = C$):

$$\begin{aligned} v_{y,o}(k+1) &= -\delta t K_{p,C} y_o(k) + (1 - \delta t K_{d,C}) v_{y,o}(k) \\ &\quad + \delta t w_y(k) + \delta t \tilde{d}_C(k); \end{aligned} \quad (29)$$

Malicious Driver ($i = M$):

$$\begin{aligned} v_{y,o}(k+1) &= \delta t K_{p,M} x_e(k) + \delta t K_{d,M} v_{x,e}(k) \\ &\quad - \delta t K_{p,M} y_o(k) + (1 - \delta t K_{d,M}) v_{y,o}(k) \\ &\quad + \delta t w_y(k) + \delta t \tilde{d}_M(k); \end{aligned} \quad (30)$$

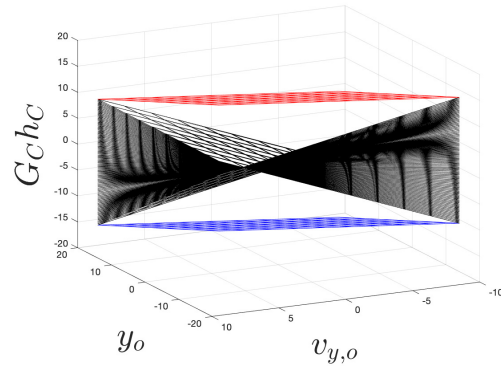


Fig. 1: Affine abstraction of the uncertain dynamics of **cautious** driver's velocity in $(y_o, v_{y,o})$ plane of projection when $w_y(k) = 0$ and $d_C(k) = 0$. The red and the blue hyperplanes are upper and lower hyperplanes obtained from abstraction, while the uncertain set is the meshed region.

Inattentive Driver ($i = I$):

$$v_{y,o}(k+1) = v_{y,o}(k) + \delta t w_y(k) + \delta t d_I(k). \quad (31)$$

Moreover, we assume that the ego car's position is constrained to be between $[0, 18]$ m at all times, and its velocity is between $[0, 9]$ m/s. The other car's position is between $[-18, 18]$ m, while its velocity is between $[-9, 9]$ m/s. The process noise signals are also bounded with a range of $[-0.01, 0.01]$.

B. Affine Abstraction

Since the uncertain PD parameters only affect the other car's velocity $v_{y,o}$ with cautious or malicious intention, we apply the proposed affine abstraction method to the uncertain functions (29) and (30), respectively.

For the cautious driver ($i = C$), the uncertain function (29) can be rewritten using the following compact form:

$$v_{y,o}(k+1) = G_C h_C(k), \quad (32)$$

where $G_C = [-\delta t K_{p,C} \quad 1 - \delta t K_{d,C} \quad \delta t \quad \delta t] \in \mathbb{R}^{1 \times 4}$ and $h_C(k) = [y_o(k) \quad v_{y,o}(k) \quad w_y(k) \quad \tilde{d}_C(k)]^T \in \mathbb{R}^4$. Using the proposed affine abstraction with state constraints and uncertain sets of intention parameters defined previously, two hyperplanes for the cautious intention are obtained as:

$$\begin{aligned} \bar{G}_C &= [-0.27 \quad 0.175 \quad 0.3 \quad 0.3], \quad \bar{b}_C = 12.285, \\ \underline{G}_C &= [-0.27 \quad 0.175 \quad 0.3 \quad 0.3], \quad \underline{b}_C = -12.285. \end{aligned}$$

As shown in Figure 1, the obtained upper and lower hyperplanes over-approximate the uncertain linear dynamics with minimum approximation error.

As illustrated in (30), all four state variables are involved in the function governing the other car's velocity with malicious intention. The compact form of (30) is given by

$$v_{y,o}(k+1) = G_M h_M(k), \quad (33)$$

where $G_M = [\delta t K_{p,M} \quad \delta t K_{d,M} \quad -\delta t K_{p,M} \quad 1 - \delta t K_{d,M} \quad \delta t \quad \delta t] \in \mathbb{R}^{1 \times 6}$ and $h_M(k) = [x_e(k) \quad v_{x,e}(k) \quad y_o(k) \quad v_{y,o}(k) \quad w_y(k) \quad \tilde{d}_C(k)]^T \in \mathbb{R}^6$. Using the same state domain and the given uncertainty sets,

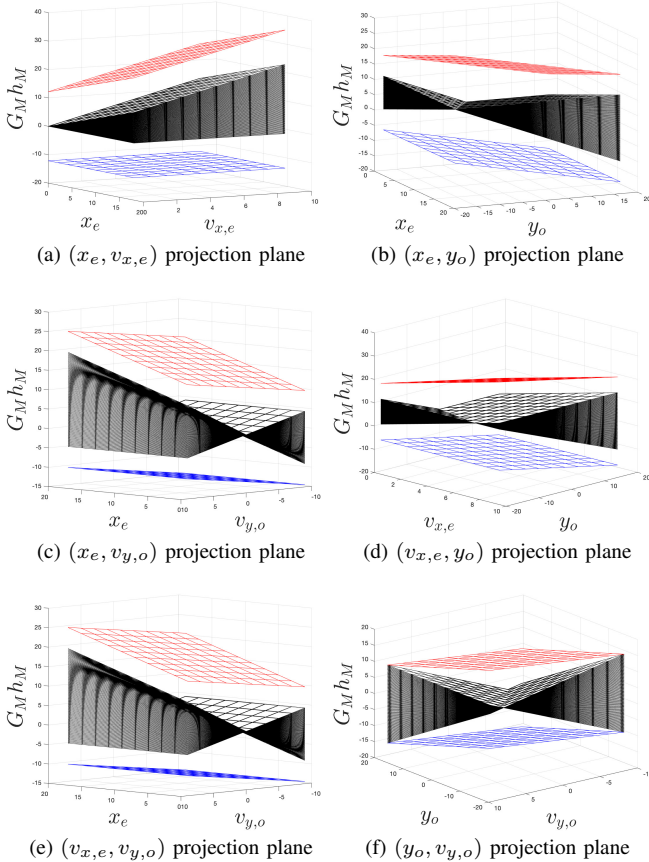


Fig. 2: Affine abstraction of the uncertain dynamics of malicious driver's velocity in different projection planes when $w_y(k) = 0$ and $\tilde{d}_M(k) = 0$.

the affine abstraction for the malicious driver is obtained as:

$$\bar{G}_M = [0.6 \quad 1.5 \quad -0.3 \quad 0.25 \quad 0.3 \quad 0.3], \quad \bar{b}_M = 12.15,$$

$$\underline{G}_M = [0 \quad 0 \quad -0.3 \quad 0.25 \quad 0.3 \quad 0.3], \quad \underline{b}_M = -12.15.$$

It is clear from Figure 2 that the obtained upper and lower hyperplanes envelop the uncertain dynamics in all projection planes. Moreover, the minimum approximation error is achieved when we project the uncertain dynamics and affine abstraction to $(y_o, v_{y,o})$ plane, as illustrated in Figure 2f.

V. CONCLUSION

In this paper, we proposed a robust optimization based affine abstraction method for uncertain affine discrete-time systems. Two affine hyperplanes are designed as upper and lower bounds to conservatively approximate the uncertain behavior over the entire domain such that all possible system trajectories are contained between the two hyperplanes. Since the affine abstractions appears to have intractable nonlinear uncertainties upon initial inspection, we recast it into a linear robust optimization problem by only using the vertices in place of every point of the uncertainty sets. Consequently, we can compute affine abstractions for the uncertain linear systems efficiently and reliably. We demonstrated the effectiveness of our approach in simulation through an affine abstraction example of the uncertain intention model in

an intersection crossing scenario. For future work, we will apply the abstracted model to active model discrimination for identifying different intentions of vehicles in highway lane changing and intersection crossing scenarios. In addition, we will consider more complex examples to study the scalability of our proposed approach with increasing state dimensions.

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