Bounded-Error Estimator Design with Missing Data Patterns via State Augmentation

Syed M. Hassaan, Qiang Shen and Sze Zheng Yong

Abstract— **In this paper, we present a bounded-error estimator that achieves equalized recovery for discrete-time time-varying affine systems subject to missing data. By augmenting the system state estimate with a Luenberger-like observer error, we formulate the equalized recovery estimator design problem as a semi-infinite optimization problem, and leverage tools from robust optimization to solve it. Due to the design freedom introduced by the Luenberger-like observer, we can place the eigenvalues of the augmented system to desired locations, which results in a more optimal intermediate level in the equalized recovery problem than existing approaches in the literature. Furthermore, as an extension of the proposed equalized recovery estimator, we consider missing data in the estimator design, where a fixed-length language is used to specify the allowable missing data patterns. Simulation examples involving an adaptive cruise control system are given to demonstrate the equalized recovery performance of the proposed estimator.**

I. Introduction

With the rapid advancement in technology, systems are becoming more complex with passing years. Since there are certain crucial system states that cannot be directly measured/observed through system outputs, state estimators, also known as state observers, are designed to tackle this problem. A great deal of current state estimators heavily depends on the accuracy of the sensor measurement. However, as systems such as autonomous vehicles, power grids, smart buildings, etc, become integrated and distributed, significant missing data or communication delays across the sensor networks may be inevitable and these issues need to be addressed when designing estimators. Otherwise, these data losses may deteriorate the estimator performance and cause the resulting closed-loop system to be unstable.

Literature review: Over the years, many different estimation techniques such as the Kalman filter [1] and Luenberger observer [2] along with their variations and extensions have been introduced. In addition to these asymptotic estimation approaches, set-valued observers [3] and ℓ_∞ filters [4], which can construct a set of compatible state values based on measured outputs, have also received considerable attention in the context of fault detection and estimation, attack identification, resilient control, etc. Taking the issues of missing or intermittent data due to sensor failures or package drops into account, several approaches for estimator design in the presence of missing data are also proposed. In [5], by modeling the arrival of the sensor data observation as a random process, a Kalman filter with missing and intermittent observations was presented. In [6], a random missing data process described by Markov chains is assumed, and a jump linear estimator is proposed, which is computationally efficient but suboptimal, to deal with missing data. However, since both approaches assume known probability distributions for the discrete state/mode switching process corresponding to missing data, they can only optimize the expected/average estimation performance. On the other hand, instead of assuming that the missing data process is stochastic, Rutledge *et al* [7] modeled the missing data by using a fixed-length language specification that specifies the set of allowable missing data patterns over a fixed time horizon. In contrast to the approaches for probabilistic intermittent observations such as [5] and [6], estimators using the fixed-length language specification for missing data patterns ensure estimation performance for the worst-case missing data scenario.

Another set of relevant literature pertains to bounded-error estimators. Recently, a new property for bounded estimation error known as equalized performance has been proposed [8], which means the estimation error will not increase at all times. In [9], a locally superstable observer with equalized performance was introduced to obtain state estimates with bounded errors from partial state observation, which further enables the synthesis of output-feedback control laws for discrete-time piecewise-affine systems subject to linear temporal logic specifications. In [7], a finite horizon affine estimator was proposed by leveraging Q-parameterization from [10] to achieve equalized recovery, i.e., the estimation error may satisfy a more relaxed error bound for a finite horizon after which it recovers to the desired error bound.

Contribution: In this paper, we propose a bounded-error dynamic estimator based on state augmentation that achieves equalized recovery of the state estimation error for discrete-time affine time-varying systems in the presence of missing data caused by sensor failures or packet drops. First, in the case of no missing data, we introduce a Luenberger-like observer to estimate the state error dynamics, and further augment our state estimate with the Luenberger-like observer error. For this augmented system, inspired by the result in [11] for affine feedback receding horizon control, we show that the equalized recovery estimator design problem can be formulated as a semi-infinite optimization problem. By

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leveraging tools from robust optimization, we can recast the equalized recovery estimator design problem as a non-convex but sparse optimization problem for which efficient off-the-shelf optimization softwares are available. Then, by expressing missing data patterns as fixed-length language specifications, we extend the proposed estimator to handle the worst-case missing data scenario (and thus, the estimator also applies for less severe missing data scenarios). Comparing with the recent bounded-error estimator in [7], since we incorporate a Luenberger-like observer into our proposed equalized recovery estimator, we have more design freedom for potentially obtaining a smaller intermediate estimation error level, as illustrated in our simulation examples. Moreover, we discuss the effects of fixing some decision variables of our non-convex formulation to obtain a convex optimization problem and provide some examples where some suitable fixed choices of these variables lead to no loss in optimality.

II. Problem Formulation

A. Notations

Throughout the paper, $\mathbb{R}^n$ is used to represent the n-dimensional Euclidean space, $\mathbb{N}$ for positive integers and $\mathbb{B}^n$ for the n-dimensional binary vector. The symbol $\otimes$ denotes the Kronecker product, $\|\cdot\|$ is used to denote the infinity norm of vectors or matrices. An identity matrix of size s is denoted by I_s, a vector of ones of length s is denoted by $\mathbb{1}_s$, while a zero matrix of dimension a-by-b is denoted by $0_{a\times b}$. The inequalities for comparing vectors and matrices are all element-wise.

B. System Dynamics

In this paper, we consider discrete-time affine time-varying systems with bounded errors as follows:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k + W_k w_k + f, \\ y_k &= \begin{cases} C_k x_k + V_k v_k, & q_k = 1, \\ \emptyset, & q_k = 0, \end{cases} \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the system state at time k , $u_k \in \mathbb{R}^m$ is the input to the system, $w_k \in \mathbb{R}^n$ is the process noise, $y_k \in \mathbb{R}^p$ is the output of the system accessible by sensors and $v_k \in \mathbb{R}^p$ is the measurement noise. The system matrices A_k, B_k, C_k, W_k, V_k and f are all known. $q_k \in \{0, 1\}$ is the measurement mode which indicates whether the measurement at time k is available or missing. We assume that w_k and v_k are bounded with $\|w_k\| \le \eta_w$ and $\|v_k\| \le \eta_v$ for every k. Without loss of generality, we assume that the initial time is $k = 0$.

Missing Data Language: As in [7], we consider a missing data model with no assumption that the measurement mode sequence follows a known stochastic process, but that they are restricted to a set of missing data patterns expressed by fixed-length language specifications, e.g., 'every n-th observation is missing' or 'at least m available measurements in the fixed interval'. More formally, within a finite time horizon T, our missing data model is a fixed-length language $\mathcal{L} \subseteq \mathbb{B}^T$ that specifies the set of allowable measurement mode sequences $\{q_k\}_{k=0}^{T-1}$.

C. Equalized Recovery

The focus of our paper is to design a bounded-error estimator, where the estimation error is guaranteed to return/recover to the same bound that it started with after a fixed number of time steps, as an extension of the notion of *equalized performance* in [8]. In terms of a time horizon T, we enforce that the estimation error bound at the end of the horizon is guaranteed to be at most the same as the bound at the start of the horizon. Formally, we consider the *equalized recovery* problem as defined in [7]:

Definition 1 (Equalized Recovery [7])**.** *An estimator is said to achieve an equalized recovery level μ_1 at time 0 with recovery time T and intermediate level $\mu_2 \ge \mu_1$ if for any initial state estimate $\hat{x}_0$ that satisfies $\|\tilde{x}_0\| \le \mu_1$, we have $\|\tilde{x}_k\| \le \mu_2$ for all $k \in [0, T]$ and $\|\tilde{x}_T\| \le \mu_1$, where $\hat{x}_k$ is the state estimate at time k and $\tilde{x}_k \triangleq x_k - \hat{x}_k$ is the estimation error.*

D. Problem Statement

The objective of this paper is to design a bounded-error estimator that satisfies *equalized recovery*, which can be stated as follows:

Problem 1 (Estimator Design)**.** *Given the system dynamics in (1), a desired recovery level μ_1, a recovery time T as a time horizon and a missing data model specified by a language $\mathcal{L}$ as well as an initial state estimate $\hat{x}_0$ that satisfies $\|\tilde{x}_0\| \le \mu_1$, design an optimal bounded-error state estimator with estimate $\hat{x}_k$ and estimation error $\tilde{x}_k = x_k - \hat{x}_k$ for all $k \in [0, T]$ that minimizes the intermediate level μ_2 subject to $\mu_2 \ge \mu_1$, $\|\tilde{x}_k\| \le \mu_2$, $\forall k \in [0, T]$ and $\|\tilde{x}_T\| \le \mu_1$.*

III. Estimator Design Approach

In order to tackle Problem 1, we consider a finite horizon dynamic estimator with an augmented state $\bar{x}_k \triangleq \begin{bmatrix} \hat{x}_k^\top & s_k^\top \end{bmatrix}^\top$ (inspired by [11]) as follows:

$$\begin{aligned} \hat{x}_{k+1} &= A_k \hat{x}_k + B_k u_k - u_{e,k} + f, \\ s_{k+1} &= A_k s_k + u_{e,k} + L_k(\tilde{y}_k - C_k s_k), \\ \hat{y}_k &= C_k \hat{x}_k, \end{aligned} \tag{2}$$

where $\hat{x}_k \in \mathbb{R}^n$ is the estimate of the system state x_k, $s_k \in \mathbb{R}^n$ is an auxiliary state (that can be interpreted as the Luenberger-like observer error), $\hat{y}_k$ is the estimated output from the estimated state, and $u_{e,k} \in \mathbb{R}^n$ is the following *causal* output error injection term:

$$u_{e,k} = \nu_k + \sum_{i=0}^{k} M_{(k,i)}(\tilde{y}_i - C_i s_i), \tag{3}$$

where $\tilde{y}_k \triangleq y_k - \hat{y}_k = y_k - C_k \hat{x}_k$ while $L_k \in \mathbb{R}^{n\times p}$, $M_{(k,i)} \in \mathbb{R}^{n\times p}$ and $\nu_k \in \mathbb{R}^n$ are to-be-designed gain matrices at time k. $\hat{x}_0$ is given whereas s_0 is a design variable. Note that unlike [11], we do not assume that $(A_k - L_k C_k)$ is stable with the Luenberger-like observer gain L_k. The effect of eigenvalues of $(A_k - L_k C_k)$, which can be controlled by L_k will be covered in the discussion in Section IV-C.

First, we present the estimator design for the perfect case when there is no missing data. Then, we will extend those results from the perfect case to the case of missing data patterns that satisfy a given fixed-length language.

A. Perfect Case: No Missing Data

In this case, we assume that the observer has access to all the measurement data at any given time, which means that $q_k = 1$ for all $k \in [0, T-1]$. Before proceeding, we define some additional notations for stacked versions of the various signals (with $e_k \triangleq \tilde{x}_k - s_k$):

$$\tilde{x} = \begin{bmatrix}\tilde{x}_0^\top & \dots & \tilde{x}_T^\top\end{bmatrix}^\top, s = \begin{bmatrix}s_0^\top & \dots & s_T^\top\end{bmatrix}^\top, e = \begin{bmatrix}e_0^\top & \dots & e_T^\top\end{bmatrix}^\top,$$
$$u_e = \begin{bmatrix}u_{e,0}^\top & \dots & u_{e,T-1}^\top\end{bmatrix}^\top, \nu = \begin{bmatrix}\nu_0^\top & \dots & \nu_{T-1}^\top\end{bmatrix}^\top,$$
$$w = \begin{bmatrix}w_0^\top & \dots & w_{T-1}^\top\end{bmatrix}^\top, v = \begin{bmatrix}v_0^\top & \dots & v_{T-1}^\top\end{bmatrix}^\top.$$

In the following theorem, we formulate the estimator design as a semi-infinite optimization problem:

Theorem 1 (Perfect Measurement Scenario)**.** *An optimal finite-horizon affine estimator* (2) *that solves Problem 1 when there is no measurement data loss is obtained by solving the following optimization problem:*

$$\begin{aligned}&\min_{M,\nu,\mu_2,s_0,L} \quad \mu_2\\ &\textit{subject to} \quad \forall(\|w\| \le \eta_w, \|v\| \le \eta_v, \|\tilde{x}_0\| \le \mu_1) :\\ &\qquad\qquad \|\tilde{x}\| \le \mu_2, \|R_T\tilde{x}\| \le \mu_1,\\ &\qquad\qquad \tilde{x} = \Theta w + \Psi v + \Xi\tilde{x}_0 + \Upsilon s_0 + E\nu,\end{aligned} \tag{4}$$

where

$$\begin{aligned}R_T &= \begin{bmatrix}0_{n\times nT} & I_n\end{bmatrix},\\ \Theta &= (I + E(M+L)C)\Gamma W,\\ \Psi &= (E(M+L)(I - C\Gamma L) - \Gamma L)V,\\ \Xi &= (I + E(M+L)C)\Phi,\\ \Upsilon &= A - \Xi,\end{aligned} \tag{5}$$

with A, C, E, L, M, W, V, Γ and Φ defined in the Appendix.

Proof. With the estimator defined in (2) for the system in (1), the error dynamics of the system will be as follows:

$$\begin{aligned}\tilde{x}_{k+1} &= x_{k+1} - \hat{x}_{k+1},\\ &= A_k\tilde{x}_k + u_{e,k} + W_k w_k,\\ \tilde{y}_k &= C_k\tilde{x}_k + V_k v_k,\end{aligned} \tag{6}$$

where $\tilde{x}_k$ and $\tilde{y}_k$ are state estimation error and output error respectively. These error dynamics in (6) can be viewed as another dynamic system with the state now being the error $\tilde{x}_k$ and the control input to the system being $u_{e,k}$.

For this 'new' system, inspired by [11], we then consider a Luenberger-like observer for the estimator design in the following form:

$$s_{k+1} = A_k s_k + u_{e,k} + L_k(\tilde{y}_k - C_k s_k), \tag{7}$$

where s_k is the Luenberger-like estimate of error $\tilde{x}_k$ and L_k is the Luenberger-like observer gain at time k.

Considering the error system defined in (6) as well as its observer defined in (7), we will have the following estimation error of the error system:

$$e_k = \tilde{x}_k - s_k. \tag{8}$$

Since the design is incorporating a finite time horizon T, we can stack the augmented system states, measurements and errors, as well as rewrite (6), (7), (8) and (3) with the stacked matrices, to obtain the following affine equations:

$$\begin{aligned}s &= As_0 + Eu_e + EL(Ce + Vv),\\ e &= \Phi e_0 - \Gamma LVv + \Gamma Ww,\\ u_e &= M(Ce + Vv) + \nu,\\ e &= \tilde{x} - s.\end{aligned} \tag{9}$$

Here, the terms s_0 and e_0 are the initial values of s and e, respectively.

Then, we can find the estimation error $\tilde{x}$ as:

$$\begin{aligned}\tilde{x} &= e + s\\ &= \Theta w + \Psi v + \Xi\tilde{x}_0 + \Upsilon s_0 + E\nu\end{aligned} \tag{10}$$

and the $\tilde{x}_T$ value at the end of the time horizon is given by

$$\tilde{x}_T = R_T\tilde{x}.$$

Finally, based on the requirements for equalized recovery in Definition 1, we must have $\|\tilde{x}_k\| \le \mu_2$ for all $k \in [0, T]$ and $\|\tilde{x}_T\| \le \mu_1$ for the worst-case noise w, v and uncertainty in the initial state estimate $\tilde{x}_0$. ∎

B. Robustification

In the optimization problem defined in Theorem 1, we have *for all* constraints involving w, v and $\tilde{x}_0$ that are semi-infinite, which makes them not readily solvable. Leveraging ideas from robust optimization [12], [13], we robustify the problem such that we only have finitely many constraints. The robustified problem is given below:

Proposition 1 (Robust Problem for Estimation)**.** *After robustification, the problem (1) takes the following form:*

$$\min_{\substack{M,\nu,\mu_2,s_0,\\ L,\Pi_1,\Pi_2}} \quad \mu_2$$
$$\textit{subject to} \quad \Pi_1 \ge 0, \Pi_2 \ge 0,$$
$$\begin{bmatrix}\Pi_1\\ \Pi_2\end{bmatrix}\begin{bmatrix}\eta_w\mathbb{1}\\ \eta_v\mathbb{1}\\ \mu_1\mathbb{1}\end{bmatrix} \le \begin{bmatrix}\mu_2\mathbb{1}\\ \mu_1\mathbb{1}\end{bmatrix} - \begin{bmatrix}I & 0\\ -I & 0\\ 0 & I\\ 0 & -I\end{bmatrix}\begin{bmatrix}E\nu + \Upsilon s_0\\ R_T(E\nu + \Upsilon s_0)\end{bmatrix},$$
$$\begin{bmatrix}\Pi_1\\ \Pi_2\end{bmatrix}\begin{bmatrix}I & 0 & 0\\ -I & 0 & 0\\ 0 & I & 0\\ 0 & -I & 0\\ 0 & 0 & I\\ 0 & 0 & -I\end{bmatrix} = \begin{bmatrix}I & 0\\ -I & 0\\ 0 & I\\ 0 & -I\end{bmatrix}\begin{bmatrix}G\\ R_T G\end{bmatrix}, \tag{11}$$

where $G \triangleq \begin{bmatrix}\Theta & \Psi & \Xi\end{bmatrix}$ with Θ, Ψ, Ξ defined in (5)*, while Π_1 and Π_2 are dual matrix variables of appropriate dimensions.*

Proof. Since '$\forall(\|w\| \le \eta_w, \|v\| \le \eta_v, \|\tilde{x}_0\| \le \mu_1)$ such that $\|\tilde{x}\| \le \mu_2$ and $\|R_T\tilde{x}\| \le \mu_1$ hold' is equivalent to the conjunction of '$\forall(\|w\| \le \eta_w, \|v\| \le \eta_v, \|\tilde{x}_0\| \le \mu_1)$ such that $\|\tilde{x}\| \le \mu_2$ holds' and '$\forall(\|w\| \le \eta_w, \|v\| \le \eta_v, \|\tilde{x}_0\| \le \mu_1)$ such that $\|R_T\tilde{x}\| \le \mu_1$ holds,' we will only derive the robustification of the former. The latter can derived similarly.

The constraint '$\forall(\|w\| \le \eta_w, \|v\| \le \eta_v, \|\tilde{x}_0\| \le \mu_1)$ such that $\|\tilde{x}\| \le \mu_2$ holds' is equivalent to

$$\max_{\substack{\|w\|\le\eta_w,\|v\|\le\eta_v,\\ \|\tilde{x}_0\|\le\mu_1}} \begin{bmatrix}I\\ -I\end{bmatrix}(\Theta w + \Psi v + \Xi\tilde{x}_0 + \Upsilon s_0 + E\nu) \le \mu_2\mathbb{1}.$$

Since the variables in the preceding maximization appear linearly, we can simply apply duality theory from the literature of linear programming with Π_1 being the dual matrix variable and there will be no duality gap, i.e., the dualization or conversion is exact. ■

Remark 1. *The optimization problem in Proposition 1 has bilinear constraints because of Ψ and the product of Υ and s_0. However, the problem is relatively sparse, hence off-the-shelf nonlinear optimization solvers can return optimal solutions very quickly, as is demonstrated in our simulation example. On the other hand, if we fix L and s_0, the problem becomes a linear programming (LP) problem. We will provide guidelines in Section IV-C on how to choose L and s_0 without any loss of optimality.*

C. Missing Data Case

To deal with measurement data loss, Theorem 1 is further extended to the missing data scenario, where the missing data patterns are described by a fixed-length language, $\mathcal{L} \subseteq \mathbb{B}^T$, which is assumed to be known. For instance, the language $\mathcal{L} = \{\sigma \in \mathbb{B}^T | \, \sigma \text{ has at least } m \text{ 1's in } T \text{ time steps}\}$ defines the missing data pattern of 'there exists at least m available data over a time horizon T.' A specific language $\mathcal{L}$ may contain multiple words, i.e., with cardinality $|\mathcal{L}| \geq 1$, but the estimator design approach in Theorem 1 only handles the case with $|\mathcal{L}| = 1$. To cope with the case where $|\mathcal{L}| > 1$, we employ a less than or equal operator ($\preceq$) defined in [7] for two words $\sigma_1 \in \mathcal{L}$ and $\sigma_2 \in \mathcal{L}$ in a language $\mathcal{L} \subseteq \mathbb{R}^T$:

$$\sigma_1 \preceq \sigma_2 \quad \iff \quad \forall i \in [1, T] : \sigma_1[i] = 0 \implies \sigma_2[i] = 0.$$

Thus, according to the $\preceq$ operator, we can obtain a *worst-case* language $\mathcal{L}^* \triangleq \{\sigma^* \in \mathbb{B}^T\}$, where σ^* is the least upper bound of the set $\mathcal{L}$ that is computed by implementing a bit-wise logic AND operation among all words in $\mathcal{L}$ [7]. As a result, we only have to consider the missing data pattern where the observations are always available to the designer.

Theorem 2 (Estimator Design with Missing Data)**.** *A finite-horizon affine estimator, which can solve Problem 1 when there is measurement data loss defined by a fixed-length language $\mathcal{L}^* = \{\sigma^*\}$ with $|\mathcal{L}^*| = 1$, will exist if the following problem has a feasible solution:*

$$\begin{aligned} \min_{M,\nu,\mu_2,s_0,L} \quad & \mu_2 \\ \textit{subject to} \quad & \forall (\|w\| \leq \eta_w, \|v\| \leq \eta_v, \|\tilde{x}_0\| \leq \mu_1, \\ & \; i \in \{i : \sigma^*(i) = 0\}) : \\ & \|\tilde{x}\| \leq \mu_2, \|R_T \tilde{x}\| \leq \mu_1, \\ & \tilde{x} = \Theta w + \Psi v + \Xi \tilde{x}_0 + \Upsilon s_0 + E\nu, \\ & M\Lambda_i = 0, L\Lambda_i = 0 \textit{ and } \nu^\top \Lambda_i = 0, \end{aligned} \tag{12}$$

where $\Lambda_i = b_i \otimes I_p$ with $b_i \in \mathbb{R}^T$ being the i-th basis vector in $\mathbb{R}^T$, while R_T, Θ, Ψ, Ξ and Υ are defined in Theorem 1.

Proof. The proof is almost the same as the proof to Theorem 1 besides additional constraints introduced to set the i-th column of matrices M and L, and the i-th row of vector v to zero, $\forall i \in \{i : \sigma^*(i) = 0\}$ for when the data is missing. ■

The robustification of the formulation in Theorem 2 can be carried out in the exact same manner as in Proposition 1. In fact, the resulting optimization problem is the same as in Proposition 1 with the addition of $M\Lambda_i = 0$ and $L\Lambda_i = 0$ for all $i \in \{i : \sigma^*(i) = 0\}$ as constraints. Moreover, since the worst-case language $\mathcal{L}^*$ is considered in Theorem 2, the constructed estimator also applies to other fixed-length languages whose worse-case is $\mathcal{L}^*$.

D. Implementation of the Estimator

The proposed equalized recovery estimator can be utilized in several different scenarios. First, if the missing data pattern periodically repeats itself with a period of T steps, then we can 'reset' the time back to k_0 every T steps and use the same estimator with the same gain matrices L, M and ν, since the proposed estimator enforces that the estimation error bound at the end of the horizon is no more than the bound at the start of the horizon.

On the other hand, when there is no missing data for certain time intervals, we can simply use an equalized performance estimator, which is essentially the equalized recovery estimator with $T = 1$ (cf. [8] for its precise definition). In addition, we can apply the equalized performance estimator to guarantee the bounded estimation error level until a missing data is observed, after which we immediately switch to an equalized recovery estimator (with a language where the first data is missing). Then, when the estimation error returns/recovers to the previous level, the equalized performance estimator can be used again until the next time a missing data is detected.

IV. Example and Discussion

In this section, using an example of a discrete-time affine system, we demonstrate capabilities of our proposed equalized recovery estimator in handling missing data patterns described by a fixed-length language $\mathcal{L}$. In particular, we compare our simulation results with those from an existing equalized recovery estimator [7] to show the advantages of our approach. Moreover, we investigate the influence of choosing different Luenberger-like observer gain L and initial state s_0 on the estimation error guarantees.

A. Adaptive Cruise Control Example

To validate the capability of achieving equalized recovery of the proposed estimator, we consider the adaptive cruise control (ACC) system in [7], which is a discrete-time affine system model (1) with time-invariant matrices given by:

$$A = \begin{bmatrix} 0.9964 & 0 & 0 \\ -0.4991 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0.3643 \times 10^{-3} \\ -0.0911 \times 10^{-3} \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, f = \begin{bmatrix} -0.0028 \\ 0.0007 \\ 0 \end{bmatrix}, W = \begin{bmatrix} 0 \\ 0.125 \\ 0.5 \end{bmatrix}, V = I_p.$$

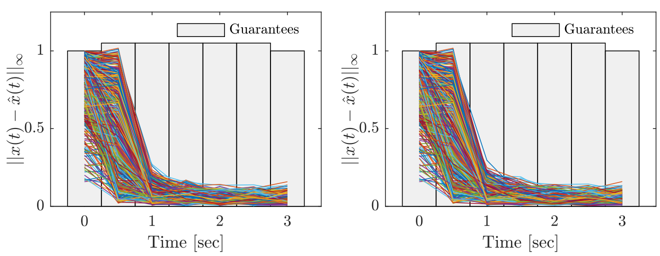
Throughout the simulation, the value of the equalized recovery level is chosen as $\mu_1 = 1$ and the time step for recovering the estimation error is specified as $T = 6$, which corresponds to a time horizon of 3 seconds.

For the simulation, we use YALMIP [14] in the MATLAB environment with various solvers to solve the problem. For the nonlinear optimization problem in Proposition 1 (when

L_k and s_0 are decision variables), we use IPOPT [15], a nonlinear programming solver, as our solver of choice because it exploits the sparsity of the matrices involved to quickly solve the problem. When we fix the values of L and s_0, the equalized estimator design problem becomes an LP problem, and in this case, we also use the Gurobi solver [16] to illustrate the influence of solver choices on the solutions.

B. Results

First, we consider the perfect scenario with no missing data. In this case, the fixed-length language is just $\mathcal{L} = \{\mathbb{1}_T\}$. When using the optimization formulation in Theorem 1, we obtain the optimal value of the intermediate level of $\mu_2 = 1.05$, which is the same as the level obtained by the estimator in [7]. It can be observed from Figure 1 that the results of both approaches (when using IPOPT) are very close to each other. Moreover, the optimizal Luenberger-like gain is $L = [1.31, -1.6; 0.05, 0.51; 0.81, -1.06]$, which results in eigenvalues of $(A-LC)$ to be $\{1.04, 0.07 \pm 0.48j\}$, which shows that $(A - LC)$ need not be stable. Moreover, the obtained $s_0 = 10^{-7}[-0.19; -0.24; 0.15]$ is evaluated by the solver to be approximately zero.



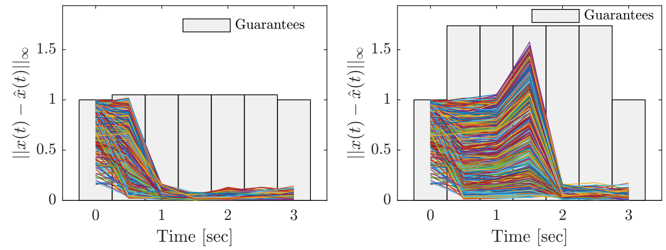
(a) Proposed Observer (b) Observer from [7]

Fig. 1: Estimation errors with no missing data for 500 runs.

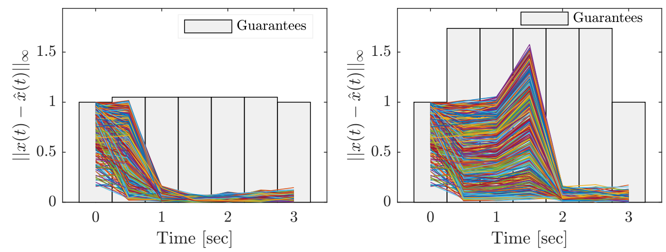
Next, we consider missing data patterns that satisfy the language $\mathcal{L} = \{101111, 110111\}$. It is clear from its definition in Section III-C that the worst-case language is $\mathcal{L}^* = \{100111\}$. Solving the optimization problem in Theorem 2 with this worst-case language, we get the same optimal value of $\mu_2 = 1.05$ with $L = [0.76, 0.04; 0.11, 1.97; 0.8, 1.97]$ that makes the eigenvalue locations to be $\lambda(A - LC) = \{-0.09, 0.18 \pm 0.07j\}$ and $s_0 = 10^{-7}[0.08; -0.1; -0.1]$ that is also close to zero. As shown in Figure 2, comparing the proposed approach with the one in [7] under the same missing data scenario, we see that, using our approach, the optimal intermediate level μ_2 remains the same as in the no missing data scenario, while the approach in [7] computes a larger μ_2. This example suggests that the proposed state augmentation approach has additional 'degrees of freedom' (with more states and gain matrices) when compared with the approach in [7], which enables our approach to obtain a smaller μ_2.

C. Discussion

Next, we discuss the effects of varying the eigenvalues of the Luenberger-like observer state matrix $A - LC$ by fixing Luenberger-like gain L and value of s_0 (such that the



(a) Proposed Observer with missing data at $k = 2$ (b) Observer from [7] with missing data at $k = 2$



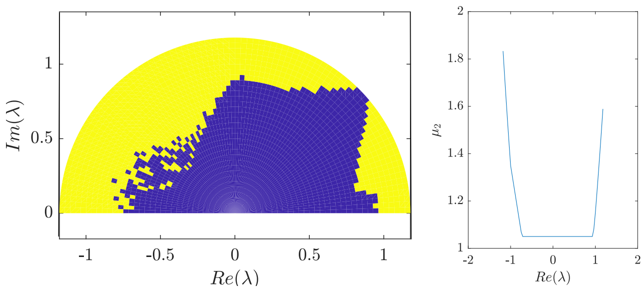
(c) Proposed Observer with missing data at $k = 3$ (d) Observer from [7] with missing data at $k = 3$

Fig. 2: Estimation errors with missing data and recovery time $T = 6$ for 500 runs (using "worst-case" language with missing data at both $k = 2$ and $k = 3$).

optimization problem in Proposition 1 becomes a linear programming (LP) problem) on the overall estimator guarantee in terms of the minimum value of μ_2.

First, to see the effect of s_0 on the estimator guarantees, we compute the optimal intermediate level μ_2 with respect to different s_0. We observed (plots omitted for brevity) that varying s_0 does not affect the optimal value of μ_2 for any chosen L. In addition, although varying s_0 results in different optimal matrices M and ν for the causal output error injection term u_e, it does not affect the resulting true estimation error $\tilde{x} = x - \hat{x}$. Thus, without any loss of optimality, we can set s_0 as a constant (e.g., $s_0 = 0$) instead of including it as a decision variable in Proposition 1.

Then, fixing $s_0 = 0$, we investigate the effect of varying the Luenberger-like gain L, and hence, the eigenvalues of the Luenberger observer error $(A - LC)$ (as an augmented new system state) on the optimal intermediate level μ_2. As before, we consider the worst-case language $\mathcal{L}^* = \{100111\}$. In particular, we vary L such that $(A - LC)$ has one eigenvalue at 0 and a pair of complex conjugate eigenvalues at $r(\cos(\theta) \pm j\sin(\theta))$ with $r \in [0, 1.5]$ and $\theta \in [0, \pi]$, and compute the optimal intermediate level μ_2 for each L. The trend obtained from solving the problem is shown in Figure 3, from which we see that the value of μ_2 does not change in the blue area in Figure 3a. Moreover, we observe the same trend for its projection onto the real axis in Figure 3b, where μ_2 does not change for real eigenvalues varying from -0.7143 to 0.9286. This implies that we can fix the Luenberger-like gain L through eigenvalue placement such that the spectral radius of $(A - LC)$ is small enough to obtain the optimal estimator guarantee instead of optimizing it, which simplifies the optimization problem in Proposition 1 to an LP problem.



(a) Heat map of μ_2 values with varying complex eigenvalue locations (Blue: $\mu_2 = 1.05$, Yellow: $\mu_2 > 1.05$).

(b) Projection of Figure 3a with $Im(\lambda) = 0$.

Fig. 3: Effects of eigenvalue location on optimal μ_2.

Finally, it is also interesting to note that the use of different solvers can result in different actual estimation error $\tilde{x}$, even though the optimal intermediate level μ_2 remains the same, as we would expect. Specifically, when we fix the values of L and s_0, we have an LP problem and both the IPOPT and Gurobi solvers can be applied. Our simulation results (not depicted for brevity) show that IPOPT yields a lower actual estimation error $\tilde{x}$ than Gurobi and the optimal values of ν and M for the two solvers are different, although their optimal intermediate levels μ_2 are identical. This observation can be attributed to the fact that Problem 1 may have multiple minimizers that result in the same optimal μ_2 (i.e, the solution is not unique), thus different solvers may select a different optimal solution that leads to different actual estimation errors $\tilde{x}$.

V. Conclusion

This paper considered the equalized recovery estimator design problem for discrete-time time-varying affine systems in the presence of missing data, where the missing data model is expressed by a fixed-length language that consists of a set of missing data patterns. First, for the case with no missing data, the system state estimate is augmented with a Luenberger-like observer error, and using this augmented system, we formulate the estimator design problem as a non-convex but sparse optimization problem. Then, an extension is presented when there is missing data, where a robust solution is provided for the worst-case language. Future work will explore the handling of communication delays and missing data when designing equalized recovery estimators.

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Appendix: Matrix Definitions

The matrices given in Theorem 1 are defined as follows:

$$W = \begin{bmatrix} W_0 & 0 & \cdots & 0 \\ 0 & W_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & W_{T-1} \end{bmatrix}, V = \begin{bmatrix} V_0 & 0 & \cdots & 0 \\ 0 & V_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & V_{T-1} \end{bmatrix},$$

$$C = \begin{bmatrix} C_0 & 0 & \cdots & 0 & 0 \\ 0 & C_1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & C_{T-1} & 0 \end{bmatrix}, L = \begin{bmatrix} L_0 & 0 & \cdots & 0 \\ 0 & L_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & L_{T-1} \end{bmatrix},$$

$$M = \begin{bmatrix} M_{(0,0)} & 0 & \cdots & 0 \\ M_{(1,0)} & M_{(1,1)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ M_{(T-1,0)} & M_{(T-1,1)} & \cdots & M_{(T-1,T-1)} \end{bmatrix},$$

$$A = \begin{bmatrix} I_n \\ A_0^1 \\ \vdots \\ A_0^{T-1} \\ A_0^T \end{bmatrix}, \ \Phi = \begin{bmatrix} I_n \\ \Phi_0^1 \\ \vdots \\ \Phi_0^{T-1} \\ \Phi_0^T \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ A_1^1 & 0 & 0 & \cdots & 0 \\ A_1^2 & A_2^2 & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ A_1^T & A_2^T & \cdots & \cdots & A_T^T \end{bmatrix}, \ \Gamma = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \Phi_1^1 & 0 & 0 & \cdots & 0 \\ \Phi_1^2 & \Phi_2^2 & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ \Phi_1^T & \Phi_2^T & \cdots & \cdots & \Phi_T^T \end{bmatrix},$$

where $A_i^k = A_{k-1}A_{k-2}...A_i$, $\Phi_i^k = \Phi_{k-1}\Phi_{k-2}...\Phi_i$ and $\Phi_k = A_k - L_kC_k$.