QUATERNION IDENTITIES

John L. Crassidis*

The purpose of this work is to maintain a collection of quaternion identities, which can be used for control and estimation purposes.

CROSS PRODUCT IDENTITIES

The matrix cross product is a skew-symmetric matrix, which is defined as

$$[\mathbf{u}\times] \equiv \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \tag{1}$$

with $\mathbf{u}\times\mathbf{v} = [\mathbf{u}\times]\mathbf{v}$. Since $[\mathbf{u}\times]\mathbf{u} = \mathbf{0}$, then $[\mathbf{u}\times]$ must be singular. The eigenvalues of $[\mathbf{u}\times]$ are given by $\lambda_1 = 0$ and $\lambda_{2,3} = \pm||\mathbf{u}||j$. Some useful identities for cross product matrix include:[1]

$$[\mathbf{u}\times]^T = -[\mathbf{u}\times] \tag{2a}$$

$$[\mathbf{u}\times]\mathbf{v} = -[\mathbf{v}\times]\mathbf{u} \tag{2b}$$

$$[\mathbf{u}\times][\mathbf{v}\times] = -\left(\mathbf{u}^T\mathbf{v}\right) I_{3\times3} + \mathbf{v}\,\mathbf{u}^T \tag{2c}$$

$$[\mathbf{u}\times]^3 = -\left(\mathbf{u}^T\mathbf{u}\right)[\mathbf{u}\times] \tag{2d}$$

$$[\mathbf{u}\times][\mathbf{v}\times] - [\mathbf{v}\times][\mathbf{u}\times] = \mathbf{v}\,\mathbf{u}^T - \mathbf{u}\,\mathbf{v}^T = [(\mathbf{u}\times\mathbf{v})\times] \tag{2e}$$

$$\mathbf{u}\,\mathbf{v}^T[\mathbf{w}\times] + [\mathbf{w}\times]\mathbf{v}\,\mathbf{u}^T = -[\{\mathbf{u}\times(\mathbf{v}\times\mathbf{w})\}\times] \tag{2f}$$

$$(I_{3\times3} - [\mathbf{u}\times])(I_{3\times3} + [\mathbf{u}\times])^{-1} = \frac{1}{1+\mathbf{u}^T\mathbf{u}}\left\{(1-\mathbf{u}^T\mathbf{u})I_{3\times3} + 2\mathbf{u}\,\mathbf{u}^T - 2[\mathbf{u}\times]\right\} \tag{2g}$$

$$||\mathbf{u}\times\mathbf{v}||^2 I_{3\times3} = \left(\mathbf{u}^T\mathbf{u}\right)\mathbf{v}\,\mathbf{v}^T + \left(\mathbf{v}^T\mathbf{v}\right)\mathbf{u}\,\mathbf{u}^T - \left(\mathbf{u}^T\mathbf{v}\right)\left(\mathbf{u}\,\mathbf{v}^T + \mathbf{v}\,\mathbf{u}^T\right) + (\mathbf{u}\times\mathbf{v})\,(\mathbf{u}\times\mathbf{v})^T \tag{2h}$$

$$\mathrm{adj}\left([\mathbf{u}\times]\right) = \mathbf{u}\,\mathbf{u}^T \tag{2i}$$

where $I_{n\times n}$ is an $n\times n$ identity matrix ($0_{n\times n}$ is an $n\times n$ matrix of zeros). Other useful properties are listed here as well. If M is an arbitrary square matrix, then

$$M[\mathbf{u}\times] + [\mathbf{u}\times]M^T + [(M^T\mathbf{u})\times] = \mathrm{Tr}(M)[\mathbf{u}\times] \tag{3a}$$

$$M[\mathbf{u}\times]M^T = [\{\mathrm{adj}(M^T)\mathbf{u}\}\times] \tag{3b}$$

$$[\{(M\mathbf{u})\times(M\mathbf{v})\}\times] = M[(\mathbf{u}\times\mathbf{v})\times]M^T \tag{3c}$$

$$(M\mathbf{u})\times(M\mathbf{v}) = \mathrm{adj}(M^T)\,(\mathbf{u}\times\mathbf{v}) \tag{3d}$$

$$[\mathbf{u}\times][\mathrm{Tr}(M)I_{3\times3} - M][\mathbf{u}\times]^T = (\mathbf{u}^T M\mathbf{u})I_{3\times3} - \mathbf{u}\,\mathbf{u}^T M^T - M^T\mathbf{u}\,\mathbf{u}^T + (\mathbf{u}^T\mathbf{u})M^T \tag{3e}$$

where Tr denotes the trace operator and adj denotes the adjoint matrix. If we write M in terms of its columns

$$M = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3\end{bmatrix} \tag{4}$$

*Professor, Department of Mechanical & Aerospace Engineering, University at Buffalo, State University of New York, Amherst, NY 14260-4400. E-mail: johnc@buffalo.edu.

then

$$\det(M) = \mathbf{u}_1^T(\mathbf{u}_2 \times \mathbf{u}_3) \tag{5}$$

where det denotes the determinant. Also, if A is an orthogonal matrix with determinant 1, then from we have Eq. (3b)

$$A[\mathbf{u}\times]A^T = [(A\mathbf{u})\times] \tag{6}$$

These cross product relations are useful to prove many of the quaternion identities shown in this document.

QUATERNION KINEMATICS

One of the most useful attitude parameterization is given by the *quaternion*.[2] Like the Euler axis/angle parameterization, the quaternion is also a four dimensional vector, defined by as

$$\mathbf{q} \equiv \begin{bmatrix} \boldsymbol{\varrho} \\ q_4 \end{bmatrix} \tag{7}$$

with

$$\boldsymbol{\varrho} \equiv \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^T = \hat{\mathbf{e}} \sin(\vartheta/2) \tag{8a}$$

$$q_4 = \cos(\vartheta/2) \tag{8b}$$

where $\hat{\mathbf{e}}$ is a unit vector corresponding to the axis of rotation and ϑ is the angle of rotation. Since a four-dimensional vector is used to describe three-dimensions, the quaternion components cannot be independent of each other. The quaternion satisfies a single constraint given by $\mathbf{q}^T\mathbf{q} = 1$, which is analogous to requiring that $\hat{\mathbf{e}}$ be a unit vector in the Euler axis/angle parameterization. Define the following matrix:

$$\begin{aligned} A(\mathbf{q}) &\equiv \Xi^T(\mathbf{q})\Psi(\mathbf{q}) \\ &= (q_4 I_{3\times 3} - [\boldsymbol{\varrho}\times])^2 + \boldsymbol{\varrho}\,\boldsymbol{\varrho}^T \\ &= (q_4^2 - \boldsymbol{\varrho}^T\boldsymbol{\varrho}) I_{3\times 3} + 2\boldsymbol{\varrho}\,\boldsymbol{\varrho}^T - 2q_4[\boldsymbol{\varrho}\times] \end{aligned} \tag{9}$$

where the matrices $\Xi(\mathbf{q})$ and $\Psi(\mathbf{q})$ are given by

$$\Xi(\mathbf{q}) \equiv \begin{bmatrix} q_4 I_{3\times 3} + [\boldsymbol{\varrho}\times] \\ -\boldsymbol{\varrho}^T \end{bmatrix} \tag{10a}$$

$$\Psi(\mathbf{q}) \equiv \begin{bmatrix} q_4 I_{3\times 3} - [\boldsymbol{\varrho}\times] \\ -\boldsymbol{\varrho}^T \end{bmatrix} \tag{10b}$$

Note that if $\mathbf{q}^T\mathbf{q} = 1$, then $A(\mathbf{q})$ is the attitude matrix. An advantage to using quaternions is that the attitude matrix is quadratic in the parameters and also does not involve transcendental functions. For small angles the vector part of the quaternion is approximately equal to half angles, so that $\boldsymbol{\varrho} \approx \boldsymbol{\alpha}/2$ and $q_4 \approx 1$, where $\boldsymbol{\alpha}$ is a vector of small angle rotations.

The quaternion kinematics equation is given by

$$\frac{d}{dt}\mathbf{q} = \frac{1}{2}\Xi(\mathbf{q})\boldsymbol{\omega} = \frac{1}{2}\Omega(\boldsymbol{\omega})\mathbf{q} \tag{11}$$

where

$$\Omega(\boldsymbol{\omega}) \equiv \begin{bmatrix} -[\boldsymbol{\omega}\times] & \boldsymbol{\omega} \\ \\ -\boldsymbol{\omega}^T & 0 \end{bmatrix} \tag{12}$$

Also, another useful identity is given by

$$\Psi(\mathbf{q})\boldsymbol{\omega} = \Gamma(\boldsymbol{\omega})\mathbf{q} \tag{13}$$

where

$$\Gamma(\boldsymbol{\omega}) \equiv \begin{bmatrix} [\boldsymbol{\omega}\times] & \boldsymbol{\omega} \\ \\ -\boldsymbol{\omega}^T & 0 \end{bmatrix} \tag{14}$$

The inverse kinematics is given by multiplying Eq. (11) by $\Xi^T(\mathbf{q})$, and using the identity in Eq. (16a), leading to

$$\boldsymbol{\omega} = 2\,\Xi^T(\mathbf{q})\,\dot{\mathbf{q}} \tag{15}$$

A major advantage of using quaternions is that the kinematics equation is linear in the quaternion and is also free of singularities.

THE MATRICES $\Xi(\mathbf{q})$, $\Psi(\mathbf{q})$, $\Omega(\boldsymbol{\omega})$ AND $\Gamma(\boldsymbol{\omega})$

The matrix $\Xi(\mathbf{q})$ obeys the following helpful relations:

$$\Xi^T(\mathbf{q})\Xi(\mathbf{q}) = (\mathbf{q}^T\mathbf{q})I_{3\times 3} \tag{16a}$$

$$\Xi(\mathbf{q})\Xi^T(\mathbf{q}) = (\mathbf{q}^T\mathbf{q})I_{4\times 4} - \mathbf{q}\,\mathbf{q}^T \tag{16b}$$

$$\Xi^T(\mathbf{q})\mathbf{q} = \mathbf{0}_{3\times 1} \tag{16c}$$

$$\Xi^T(\mathbf{q})\boldsymbol{\lambda} = -\Xi^T(\boldsymbol{\lambda})\mathbf{q} \quad \text{for any } \boldsymbol{\lambda}_{4\times 1} \tag{16d}$$

Note that $\Psi(\mathbf{q})$ also follows the same identities shown in Eq. (16). The matrices $\Omega(\boldsymbol{\omega})$ and $\Gamma(\boldsymbol{\omega})$ follow

$$\Omega^2(\boldsymbol{\omega}) = \Gamma^2(\boldsymbol{\omega}) = -(\boldsymbol{\omega}^T\boldsymbol{\omega})I_{4\times 4} \tag{17}$$

The determinants of $\Omega(\boldsymbol{\omega})$ and $\Gamma(\boldsymbol{\omega})$ are given by

$$\det[\Omega(\boldsymbol{\omega})] = \det[\Gamma(\boldsymbol{\omega})] = (\boldsymbol{\omega}^T\boldsymbol{\omega})^2 \tag{18}$$

The eigenvalues of both $\Omega(\boldsymbol{\omega})$ and $\Gamma(\boldsymbol{\omega})$ are given by $\lambda_{1,2,3,4} = \pm||\boldsymbol{\omega}||j$. Also, $\Omega(\mathbf{b})$ and $\Gamma(\mathbf{r})$ commute for any $\mathbf{b}$ and $\mathbf{r}$, so that

$$\Omega(\mathbf{b})\Gamma(\mathbf{r}) = \Gamma(\mathbf{r})\Omega(\mathbf{b}) \tag{19}$$

Equation (17) is useful to simplify the derivative of Eq. (11), with

$$\begin{aligned} \ddot{\mathbf{q}} &= \frac{1}{2}\Xi(\mathbf{q})\dot{\boldsymbol{\omega}} + \frac{1}{2}\Omega(\boldsymbol{\omega})\dot{\mathbf{q}} \\ &= \frac{1}{2}\Xi(\mathbf{q})\dot{\boldsymbol{\omega}} - \frac{1}{4}(\boldsymbol{\omega}^T\boldsymbol{\omega})\mathbf{q} \end{aligned} \tag{20}$$

where Eq. (11) has been used in Eq. (20). Note that the dynamics equation[3] for $\dot{\boldsymbol{\omega}}$ can be substituted into Eq. (20), which relates the quaternion to a torque input.

The identities in Eqs. (9), (11) and (13) are very useful in attitude determination. Consider the following identity:

$$\begin{aligned}\mathbf{b}^T A(\mathbf{q})\mathbf{r} &= \mathbf{b}^T \Xi^T(\mathbf{q})\Psi(\mathbf{q})\mathbf{r} \\ &= -\mathbf{q}^T \Omega(\mathbf{b})\Gamma(\mathbf{r})\mathbf{q}\end{aligned} \tag{21}$$

Hence, the matrix $\Omega(\mathbf{b})\Gamma(\mathbf{r})$ can used to form Davenport's K matrix:[4]

$$K \equiv \begin{bmatrix} B + B^T - \mathrm{Tr}(B) I_{3\times 3} & (\mathbf{b}\times\mathbf{r}) \\ (\mathbf{b}\times\mathbf{r})^T & \mathrm{Tr}(B) \end{bmatrix} = -\Omega(\mathbf{b})\Gamma(\mathbf{r}) \tag{22}$$

where

$$B \equiv \mathbf{b}\,\mathbf{r}^T \tag{23}$$

Another interesting approach involves writing Eq. (21) as

$$\begin{aligned}\mathbf{b}^T A(\mathbf{q})\mathbf{r} &= -\mathbf{q}^T \Omega(\mathbf{b})\Gamma(\mathbf{r})\mathbf{q} \\ &= \mathbf{q}^T \left[||\mathbf{b}|| \, ||\mathbf{r}|| I_{4\times 4} - C^T C \right] \mathbf{q}\end{aligned} \tag{24}$$

with

$$C = \frac{||\mathbf{b}||^{1/2} ||\mathbf{r}||^{1/2}}{\sqrt{2}} \begin{bmatrix} [(\hat{\mathbf{b}} + \hat{\mathbf{r}})\times] & -(\hat{\mathbf{b}} - \hat{\mathbf{r}}) \\ (\hat{\mathbf{b}} - \hat{\mathbf{r}})^T & 0 \end{bmatrix} \tag{25}$$

where $\hat{\mathbf{b}} = \mathbf{b}/||\mathbf{b}||$ and $\hat{\mathbf{r}} = \mathbf{r}/||\mathbf{r}||$. This can be used to develop a square-root attitude determination algorithm.[5] Another identity closely related to Eq. (24) is given by[6]

$$\Xi(\mathbf{q})\mathbf{b} - \Psi(\mathbf{q})\mathbf{r} = \begin{bmatrix} -[(\mathbf{b} + \mathbf{r})\times] & (\mathbf{b} - \mathbf{r}) \\ -(\mathbf{b} - \mathbf{r})^T & 0 \end{bmatrix} \mathbf{q} \tag{26}$$

Also, using specific relationships of $\mathbf{r}$ to $\mathbf{b}$, the following identities are true:[7]

$$2(\mathbf{q}^T\mathbf{q})^{-1}\Xi(\mathbf{q})[\mathbf{b}\times]\Xi^T(\mathbf{q}) = \begin{bmatrix} [(\mathbf{b} + \mathbf{r})\times] & -(\mathbf{b} - \mathbf{r}) \\ (\mathbf{b} - \mathbf{r})^T & 0 \end{bmatrix}, \text{ with } \mathbf{r} = (\mathbf{q}^T\mathbf{q})^{-1} A^T(\mathbf{q})\mathbf{b} \tag{27a}$$

$$2(\mathbf{q}^T\mathbf{q})^{-1}\Psi(\mathbf{q})[\mathbf{b}\times]\Psi^T(\mathbf{q}) = \begin{bmatrix} [(\mathbf{b} + \mathbf{r})\times] & (\mathbf{b} - \mathbf{r}) \\ -(\mathbf{b} - \mathbf{r})^T & 0 \end{bmatrix}, \text{ with } \mathbf{r} = (\mathbf{q}^T\mathbf{q})^{-1} A(\mathbf{q})\mathbf{b} \tag{27b}$$

Equations (26) and (27) are derived from a measurement model that is linear in the quaternion.

Some useful identities are given by[1]

$$\Omega(\boldsymbol{\omega})\Xi(\mathbf{q}) = -\Xi(\mathbf{q})[\boldsymbol{\omega}\times] - \mathbf{q}\,\boldsymbol{\omega}^T \tag{28a}$$

$$\Gamma(\boldsymbol{\omega})\Psi(\mathbf{q}) = \Psi(\mathbf{q})[\boldsymbol{\omega}\times] - \mathbf{q}\,\boldsymbol{\omega}^T \tag{28b}$$

Other identities are given by

$$\Omega(\boldsymbol{\omega})\Psi(\mathbf{q}) = [-q_4 I_{4\times4} + \Omega(\boldsymbol{\varrho})]\begin{bmatrix}[\boldsymbol{\omega}\times]\\ \\ \boldsymbol{\omega}^T\end{bmatrix} - \begin{bmatrix}2(\boldsymbol{\varrho}^T\boldsymbol{\omega})I_{3\times3}\\ \\ \mathbf{0}_{3\times1}^T\end{bmatrix} \tag{29a}$$
$$= -(\mathbf{q}^T\mathbf{q})^{-1}\left\{\Xi(\mathbf{q})[\boldsymbol{\omega}\times] + \mathbf{q}\,\boldsymbol{\omega}^T\right\}A(\mathbf{q})$$

$$\Gamma(\boldsymbol{\omega})\Xi(\mathbf{q}) = [-q_4 I_{4\times4} + \Gamma(\boldsymbol{\varrho})]\begin{bmatrix}-[\boldsymbol{\omega}\times]\\ \\ \boldsymbol{\omega}^T\end{bmatrix} - \begin{bmatrix}2(\boldsymbol{\varrho}^T\boldsymbol{\omega})I_{3\times3}\\ \\ \mathbf{0}_{3\times1}^T\end{bmatrix} \tag{29b}$$
$$= (\mathbf{q}^T\mathbf{q})^{-1}\left\{\Psi(\mathbf{q})[\boldsymbol{\omega}\times] - \mathbf{q}\,\boldsymbol{\omega}^T\right\}A^T(\mathbf{q})$$

More identities are given by

$$\Xi^T(\mathbf{q})\Omega(\boldsymbol{\omega})\Xi(\mathbf{q}) = -(\mathbf{q}^T\mathbf{q})[\boldsymbol{\omega}\times] \tag{30a}$$
$$\Psi^T(\mathbf{q})\Gamma(\boldsymbol{\omega})\Psi(\mathbf{q}) = (\mathbf{q}^T\mathbf{q})[\boldsymbol{\omega}\times] \tag{30b}$$
$$\Xi^T(\mathbf{q})\Gamma(\boldsymbol{\omega})\Xi(\mathbf{q}) = [A(\mathbf{q})\boldsymbol{\omega}\times] \tag{30c}$$
$$\Psi^T(\mathbf{q})\Omega(\boldsymbol{\omega})\Psi(\mathbf{q}) = -[A^T(\mathbf{q})\boldsymbol{\omega}\times] \tag{30d}$$
$$\Xi\left(\Gamma(\boldsymbol{\omega})\mathbf{q}\right) = \Gamma(\boldsymbol{\omega})\Xi(\mathbf{q}) \tag{30e}$$
$$\Psi\left(\Omega(\boldsymbol{\omega})\mathbf{q}\right) = \Omega(\boldsymbol{\omega})\Psi(\mathbf{q}) \tag{30f}$$
$$\Xi\left(\Omega(\boldsymbol{\omega})\mathbf{q}\right) = (\mathbf{q}^T\mathbf{q})^{-1}\Gamma\left(A^T(\mathbf{q})\boldsymbol{\omega}\right)\Xi(\mathbf{q}) = -\mathbf{q}\,\boldsymbol{\omega}^T + \Xi(\mathbf{q})[\boldsymbol{\omega}\times] \tag{30g}$$
$$\Psi\left(\Gamma(\boldsymbol{\omega})\mathbf{q}\right) = (\mathbf{q}^T\mathbf{q})^{-1}\Omega\left(A(\mathbf{q})\boldsymbol{\omega}\right)\Psi(\mathbf{q}) = -\left\{\mathbf{q}\,\boldsymbol{\omega}^T + \Psi(\mathbf{q})[\boldsymbol{\omega}\times]\right\} \tag{30h}$$
$$\Omega(\mathbf{b}\times\mathbf{r}) = \frac{1}{2}\left[\Omega(\mathbf{r})\Omega(\mathbf{b}) - \Omega(\mathbf{b})\Omega(\mathbf{r})\right] \tag{30i}$$
$$\Gamma(\mathbf{b}\times\mathbf{r}) = \frac{1}{2}\left[\Gamma(\mathbf{b})\Gamma(\mathbf{r}) - \Gamma(\mathbf{r})\Gamma(\mathbf{b})\right] \tag{30j}$$
$$\left[\left(\Xi^T(\mathbf{q})\,K\,\mathbf{q}\right)\times\right] = A(\mathbf{q})B^T - B\,A^T(\mathbf{q}) \tag{30k}$$
$$\Xi^T(\mathbf{q})\,K\,\mathbf{q} = [\mathbf{b}\times]A(\mathbf{q})\mathbf{r} \tag{30l}$$
$$\Psi^T(\mathbf{q})\,K\,\mathbf{q} = -[\mathbf{r}\times]A^T(\mathbf{q})\mathbf{b} \tag{30m}$$

where K and B are given by Eqs. (22) and (23), respectively. Also, Eqs. (30i)-(30m) are valid for any 3×1 vectors $\mathbf{b}$ and $\mathbf{r}$. Let's see how the identities in Eq. (30) can be used to derive the sensitivity matrix used in the extended Kalman filter.[8] Our goal is to find an expression for

$$H \equiv \frac{\partial}{\partial\mathbf{q}}\left[(\mathbf{q}^T\mathbf{q})^{-1}A(\mathbf{q})\mathbf{r}\right] = \frac{\partial}{\partial\mathbf{q}}\left[(\mathbf{q}^T\mathbf{q})^{-1}\Xi^T(\mathbf{q})\Gamma(\mathbf{r})\mathbf{q}\right] \tag{31}$$

where the identities in Eqs. (9) and (13) have been used in Eq. (31). Evaluating the partial in Eq. (31) gives

$$H = (\mathbf{q}^T\mathbf{q})^{-1}\left\{\frac{\partial}{\partial\mathbf{q}}\left[\Xi^T(\mathbf{q})\right]\right\}\Gamma(\mathbf{r})\mathbf{q} + (\mathbf{q}^T\mathbf{q})^{-1}\Xi^T(\mathbf{q})\Gamma(\mathbf{r}) - 2\,(\mathbf{q}^T\mathbf{q})^{-2}\Xi^T(\mathbf{q})\Gamma(\mathbf{r})\mathbf{q}\,\mathbf{q}^T \tag{32}$$

Using Eqs. (16d) and (30e) in Eq. (32) leads to

$$H = (\mathbf{q}^T\mathbf{q})^{-1}\Xi^T(\mathbf{q})\Gamma(\mathbf{r}) + (\mathbf{q}^T\mathbf{q})^{-1}\Xi^T(\mathbf{q})\Gamma(\mathbf{r}) - 2\,(\mathbf{q}^T\mathbf{q})^{-2}\Xi^T(\mathbf{q})\Gamma(\mathbf{r})\mathbf{q}\,\mathbf{q}^T \tag{33}$$

Assuming that $\mathbf{q}^T\mathbf{q} = 1$ and collecting terms yields

$$H = 2\,\Xi^T(\mathbf{q})\Gamma(\mathbf{r})\left[I_{4\times 4} - \mathbf{q}\,\mathbf{q}^T\right] = 2\,\Xi^T(\mathbf{q})\Gamma(\mathbf{r})\Xi(\mathbf{q})\Xi^T(\mathbf{q}) \tag{34}$$

where Eq. (16b) has been used in Eq. (34). Using Eq. (30c) in Eq. (34) gives

$$H = 2\,[A(\mathbf{q})\mathbf{r}\times]\,\Xi^T(\mathbf{q}) \tag{35}$$

Note that the matrix $[A(\mathbf{q})\mathbf{r}\times]$ is used in the multiplicative filter.[8]

SUCCESSIVE ROTATIONS

Another advantage of quaternions is that successive rotations can be accomplished using quaternion multiplication. Here we adopt the convention of Lefferts, Markley, and Shuster[8] who multiply the quaternions in the same order as the attitude matrix multiplication (in contrast to the usual convention established by Hamilton[2]). Suppose we wish to perform a successive rotation. This can be written using

$$A(\bar{\mathbf{q}})A(\mathbf{q}) = A(\bar{\mathbf{q}} \otimes \mathbf{q}) \tag{36}$$

The composition of the quaternions is bilinear, with[1]

$$\bar{\mathbf{q}} \otimes \mathbf{q} = Q_L(\bar{\mathbf{q}})\,\mathbf{q} = Q_R(\mathbf{q})\,\bar{\mathbf{q}} \tag{37}$$

where

$$Q_R(\mathbf{q}) \equiv \begin{bmatrix} \Xi(\mathbf{q}) & \mathbf{q} \end{bmatrix} \tag{38a}$$

$$Q_L(\mathbf{q}) \equiv \begin{bmatrix} \Psi(\mathbf{q}) & \mathbf{q} \end{bmatrix} \tag{38b}$$

Note that $Q_R(\mathbf{q})$ and $Q_L(\mathbf{q})$ are orthogonal matrices if $\mathbf{q}^T\mathbf{q} = 1$, so that

$$Q_R(\mathbf{q})Q_R^T(\mathbf{q}) = (\mathbf{q}^T\mathbf{q})I_{4\times 4} \tag{39a}$$

$$Q_L(\mathbf{q})Q_L^T(\mathbf{q}) = (\mathbf{q}^T\mathbf{q})I_{4\times 4} \tag{39b}$$

The matrices $\Omega(\boldsymbol{\omega})$ and $\Gamma(\boldsymbol{\omega})$ can be written as

$$\Omega(\boldsymbol{\omega}) = Q_L\left(\begin{bmatrix} \boldsymbol{\omega} \\ 0 \end{bmatrix}\right) \tag{40a}$$

$$\Gamma(\boldsymbol{\omega}) = Q_R\left(\begin{bmatrix} \boldsymbol{\omega} \\ 0 \end{bmatrix}\right) \tag{40b}$$

Two identities are given as follows:

$$Q_L(\mathbf{q})\,\Gamma(\boldsymbol{\omega}) = \Gamma(\boldsymbol{\omega})\,Q_L(\mathbf{q}) \tag{41a}$$

$$Q_R(\mathbf{q})\,\Omega(\boldsymbol{\omega}) = \Omega(\boldsymbol{\omega})\,Q_R(\mathbf{q}) \tag{41b}$$

The conjugate quaternion is given by

$$\mathbf{q}^* \equiv \begin{bmatrix} -\boldsymbol{\varrho} \\ q_4 \end{bmatrix} = T\,\mathbf{q} \tag{42}$$

which is directly related to the complex conjugate,[9] where

$$T \equiv \begin{bmatrix} -I_{3\times 3} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{3\times 1}^T & 1 \end{bmatrix} \tag{43}$$

Note that $T^{-1} = T$. The inverse quaternion is defined by

$$\mathbf{q}^{-1} = \frac{\mathbf{q}^*}{||\mathbf{q}||} \tag{44}$$

Note that $\mathbf{q} \otimes \mathbf{q}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T \equiv \mathbf{I}_q^T$ if $\mathbf{q}^T\mathbf{q} = 1$, which is the identity quaternion (if $\mathbf{q}$ is not normalized then $\mathbf{q} \otimes \mathbf{q}^* = \begin{bmatrix} 0 & 0 & 0 & (\mathbf{q}^T\mathbf{q}) \end{bmatrix}^T$). The quaternion conjugate and inverse are very useful since

$$A^{-1}(\mathbf{q}/||\mathbf{q}||) = A^T(\mathbf{q}/||\mathbf{q}||) = A\left(\mathbf{q}^*/||\mathbf{q}^*||\right) \tag{45}$$

The quaternion and its conjugate also follow

$$\mathbf{q} = Q_L(\mathbf{q})\mathbf{I}_q = Q_R(\mathbf{q})\mathbf{I}_q \tag{46a}$$
$$\mathbf{q}^* = Q_L(\mathbf{q}^*)\mathbf{I}_q = Q_R(\mathbf{q}^*)\mathbf{I}_q \tag{46b}$$

Some useful identities are given by

$$Q_R(\mathbf{q}^*) = Q_R^T(\mathbf{q}) \tag{47a}$$
$$Q_L(\mathbf{q}^*) = Q_L^T(\mathbf{q}) \tag{47b}$$

and

$$T\,Q_L(\mathbf{q}) = Q_R(\mathbf{q}^*)\,T \tag{48a}$$
$$Q_L(\mathbf{q})\,T = T\,Q_R(\mathbf{q}^*) \tag{48b}$$

Other useful identities are given by[1]

$$[T\,Q_L(\mathbf{q})]^2 = [Q_L(\mathbf{q})\,T]^2 = Q_L(\mathbf{q})Q_R(\mathbf{q}^*) = Q_R(\mathbf{q}^*)Q_L(\mathbf{q}) = \begin{bmatrix} A(\mathbf{q}) & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{3\times 1}^T & (\mathbf{q}^T\mathbf{q}) \end{bmatrix} \tag{49a}$$

$$Q_L(\mathbf{q}^*)\Xi(\mathbf{q})A(\mathbf{q}) = (\mathbf{q}^T\mathbf{q})^2 \begin{bmatrix} I_{3\times 3} \\ \mathbf{0}_{3\times 1}^T \end{bmatrix} \tag{49b}$$

$$Q_R(\mathbf{q}^*)\Psi(\mathbf{q})A^T(\mathbf{q}) = (\mathbf{q}^T\mathbf{q})^2 \begin{bmatrix} I_{3\times 3} \\ \mathbf{0}_{3\times 1}^T \end{bmatrix} \tag{49c}$$

More identities are given by

$$Q_L(\mathbf{q})Q_R(\bar{\mathbf{q}}) = Q_R(\bar{\mathbf{q}})Q_L(\mathbf{q}) \tag{50a}$$
$$Q_R(\mathbf{I}_q) = Q_L(\mathbf{I}_q) = I_{4\times 4} \tag{50b}$$
$$Q_L(\bar{\mathbf{q}} \otimes \mathbf{q}) = Q_L(\bar{\mathbf{q}})Q_L(\mathbf{q}) \tag{50c}$$
$$Q_R(\bar{\mathbf{q}} \otimes \mathbf{q}) = Q_R(\mathbf{q})Q_R(\bar{\mathbf{q}}) \tag{50d}$$
$$Q_L^{-1}(\mathbf{q}) = (\mathbf{q}^T\mathbf{q})^{-1}Q_L(\mathbf{q}^*) \tag{50e}$$
$$Q_R^{-1}(\mathbf{q}) = (\mathbf{q}^T\mathbf{q})^{-1}Q_R(\mathbf{q}^*) \tag{50f}$$
$$Q_L(\mathbf{q})\Omega(\mathbf{r})Q_L(\mathbf{q}^*) = \Omega\left(A(\mathbf{q})\mathbf{r}\right) \tag{50g}$$
$$Q_R(\mathbf{q}^*)\Gamma(\mathbf{r})Q_R(\mathbf{q}) = \Gamma\left(A(\mathbf{q})\mathbf{r}\right) \tag{50h}$$

and

$$\begin{aligned} Q_L(\mathbf{q} \otimes \bar{\mathbf{q}}) &= (\mathbf{q}^T\mathbf{q})^{-1}Q_L(\mathbf{q})Q_L(\bar{\mathbf{q}} \otimes \mathbf{q})Q_L^T(\mathbf{q}) \\ &= (\bar{\mathbf{q}}^T\bar{\mathbf{q}})^{-1}Q_L^T(\bar{\mathbf{q}})Q_L(\bar{\mathbf{q}} \otimes \mathbf{q})Q_L(\bar{\mathbf{q}}) \end{aligned} \tag{51a}$$
$$\begin{aligned} Q_R(\mathbf{q} \otimes \bar{\mathbf{q}}) &= (\mathbf{q}^T\mathbf{q})^{-1}Q_R^T(\mathbf{q})Q_R(\bar{\mathbf{q}} \otimes \mathbf{q})Q_R(\mathbf{q}) \\ &= (\bar{\mathbf{q}}^T\bar{\mathbf{q}})^{-1}Q_R(\bar{\mathbf{q}})Q_R(\bar{\mathbf{q}} \otimes \mathbf{q})Q_R^T(\bar{\mathbf{q}}) \end{aligned} \tag{51b}$$

A common quantity used in estimation and control is the error quaternion between two quaternions, denoted by

$$\boldsymbol{\delta}\mathbf{q} \equiv \begin{bmatrix} \boldsymbol{\delta\varrho} \\ \delta q_4 \end{bmatrix} = \mathbf{q} \otimes \bar{\mathbf{q}}^* \tag{52}$$

where $\bar{\mathbf{q}}$ is the estimated quaternion in estimation theory or the desired quaternion in control theory. Using the rules of quaternion multiplication, $\boldsymbol{\delta\varrho}$ and δq_4 can be shown to be given by

$$\boldsymbol{\delta\varrho} = \Xi^T(\bar{\mathbf{q}})\mathbf{q} \tag{53a}$$
$$\delta q_4 = \bar{\mathbf{q}}^T\mathbf{q} \tag{53b}$$

Note that as $\bar{\mathbf{q}}$ approaches $\mathbf{q}$, then $\boldsymbol{\delta\varrho}$ approaches zero. Another error quaternion is defined by

$$\boldsymbol{\delta}\mathbf{q}_{\mathcal{I}} \equiv \begin{bmatrix} \boldsymbol{\delta\varrho}_{\mathcal{I}} \\ \delta q_4 \end{bmatrix} = \bar{\mathbf{q}}^* \otimes \mathbf{q} = (\bar{\mathbf{q}}^T\bar{\mathbf{q}})^{-1}(\bar{\mathbf{q}}^* \otimes \boldsymbol{\delta}\mathbf{q} \otimes \bar{\mathbf{q}}) \tag{54a}$$
$$\boldsymbol{\delta}\mathbf{q} = (\bar{\mathbf{q}}^T\bar{\mathbf{q}})^{-1}(\bar{\mathbf{q}} \otimes \boldsymbol{\delta}\mathbf{q}_{\mathcal{I}} \otimes \bar{\mathbf{q}}^*) \tag{54b}$$

where $\boldsymbol{\delta}\mathbf{q}_{\mathcal{I}}$ is used to denote a "space-referenced error quaternion" vector. Using the rules of quaternion multiplication, $\boldsymbol{\delta\varrho}_{\mathcal{I}}$ can be shown to be given by

$$\boldsymbol{\delta\varrho}_{\mathcal{I}} = \Psi^T(\bar{\mathbf{q}})\mathbf{q} \tag{55}$$

Relationships between $\boldsymbol{\delta\varrho}$ and $\boldsymbol{\delta\varrho}_{\mathcal{I}}$ are given by

$$\boldsymbol{\delta\varrho} = (\mathbf{q}^T\mathbf{q})^{-1}A(\mathbf{q})\boldsymbol{\delta\varrho}_{\mathcal{I}} = (\bar{\mathbf{q}}^T\bar{\mathbf{q}})^{-1}A(\bar{\mathbf{q}})\boldsymbol{\delta\varrho}_{\mathcal{I}} \tag{56a}$$
$$\boldsymbol{\delta\varrho}_{\mathcal{I}} = (\mathbf{q}^T\mathbf{q})^{-1}A^T(\mathbf{q})\boldsymbol{\delta\varrho} = (\bar{\mathbf{q}}^T\bar{\mathbf{q}})^{-1}A^T(\bar{\mathbf{q}})\boldsymbol{\delta\varrho} \tag{56b}$$

Equation (56) clearly shows the meaning of the space-referenced error quaternion. Identities involving $\Xi^T(\bar{\mathbf{q}})\Xi(\mathbf{q})$ and $\Psi^T(\bar{\mathbf{q}})\Psi(\mathbf{q})$ are given by[10]

$$\Xi^T(\bar{\mathbf{q}})\Xi(\mathbf{q}) = \delta q_4 I_{3\times3} + [\boldsymbol{\delta\varrho}\times] \tag{57a}$$

$$\left[\Xi^T(\bar{\mathbf{q}})\Xi(\mathbf{q})\right]^{-1} = (\boldsymbol{\delta}\mathbf{q}^T\boldsymbol{\delta}\mathbf{q})^{-1}\left[\delta q_4 I_{3\times3} - [\boldsymbol{\delta\varrho}\times] + \frac{\boldsymbol{\delta\varrho}\,\boldsymbol{\delta\varrho}^T}{\delta q_4}\right] \tag{57b}$$

$$\Psi^T(\bar{\mathbf{q}})\Psi(\mathbf{q}) = \delta q_4 I_{3\times3} - [\boldsymbol{\delta\varrho}_{\mathcal{I}}\times] \tag{57c}$$

$$\left[\Psi^T(\bar{\mathbf{q}})\Psi(\mathbf{q})\right]^{-1} = (\boldsymbol{\delta}\mathbf{q}_{\mathcal{I}}^T\boldsymbol{\delta}\mathbf{q}_{\mathcal{I}})^{-1}\left[\delta q_4 I_{3\times3} + [\boldsymbol{\delta\varrho}_{\mathcal{I}}\times] + \frac{\boldsymbol{\delta\varrho}_{\mathcal{I}}\boldsymbol{\delta\varrho}_{\mathcal{I}}^T}{\delta q_4}\right] \tag{57d}$$

$$\left[\Xi^T(\bar{\mathbf{q}})\Xi(\mathbf{q})\right]^{-1}\Xi^T(\mathbf{q})\Xi(\bar{\mathbf{q}}) = (\mathbf{q}^T\mathbf{q})^{-1}(\bar{\mathbf{q}}^T\bar{\mathbf{q}})^{-1}A(\mathbf{q})A^T(\bar{\mathbf{q}}) \tag{57e}$$

$$\left[\Psi^T(\bar{\mathbf{q}})\Psi(\mathbf{q})\right]^{-1}\Psi^T(\mathbf{q})\Psi(\bar{\mathbf{q}}) = (\mathbf{q}^T\mathbf{q})^{-1}(\bar{\mathbf{q}}^T\bar{\mathbf{q}})^{-1}A^T(\mathbf{q})A(\bar{\mathbf{q}}) \tag{57f}$$

Note that the inverses of $\Xi^T(\bar{\mathbf{q}})\Xi(\mathbf{q})$ and $\Psi^T(\bar{\mathbf{q}})\Psi(\mathbf{q})$ are singular for 180 degree errors. Equation (57) can used to develop a control law that produces linear error dynamics.[10,11] Some other identities involving these inverses are given by

$$\left[\Xi^T(\bar{\mathbf{q}})\Xi(\mathbf{q})\right]^{-1}\boldsymbol{\delta\varrho} = \frac{\boldsymbol{\delta\varrho}}{\delta q_4} \tag{58a}$$

$$\left[\Psi^T(\bar{\mathbf{q}})\Psi(\mathbf{q})\right]^{-1}\boldsymbol{\delta\varrho}_{\mathcal{I}} = \frac{\boldsymbol{\delta\varrho}_{\mathcal{I}}}{\delta q_4} \tag{58b}$$

$$2\left[\Xi^T(\bar{\mathbf{q}})\Xi(\mathbf{q})\right]^{-1}\Xi^T(\dot{\bar{\mathbf{q}}})\mathbf{q} = -(\mathbf{q}^T\mathbf{q})^{-1}(\bar{\mathbf{q}}^T\bar{\mathbf{q}})^{-1}\delta A\,\bar{\boldsymbol{\omega}} \tag{58c}$$

$$2\left[\Xi^T(\bar{\mathbf{q}})\Xi(\mathbf{q})\right]^{-1}\Xi^T(\dot{\bar{\mathbf{q}}})\Omega(\boldsymbol{\omega})\mathbf{q} = (\mathbf{q}^T\mathbf{q})^{-1}(\bar{\mathbf{q}}^T\bar{\mathbf{q}})^{-1}\left\{[\boldsymbol{\omega}\times]\delta A\,\bar{\boldsymbol{\omega}} + \frac{\boldsymbol{\omega}^T\delta A\,\bar{\boldsymbol{\omega}}}{\delta q_4}\boldsymbol{\delta\varrho}\right\} \tag{58d}$$

$$2\left[\Xi^T(\bar{\mathbf{q}})\Xi(\mathbf{q})\right]^{-1}\Xi^T(\ddot{\bar{\mathbf{q}}})\mathbf{q} = -(\mathbf{q}^T\mathbf{q})^{-1}(\bar{\mathbf{q}}^T\bar{\mathbf{q}})^{-1}\delta A\,\dot{\bar{\boldsymbol{\omega}}} - \frac{\bar{\boldsymbol{\omega}}^T\bar{\boldsymbol{\omega}}}{2\delta q_4}\boldsymbol{\delta\varrho} \tag{58e}$$

where $\delta A \equiv A(\mathbf{q})A^T(\bar{\mathbf{q}})$. Note that if the quaternions are normalized, then Eqs. (58a) and (58b) are each equivalent to a *Rodrigues vector* or *Gibbs vector*.

Other relations involving the quaternion conjugate are given by[1]

$$\begin{bmatrix} A(\mathbf{q})\boldsymbol{\omega} \\ 0 \end{bmatrix} = \mathbf{q}\otimes\begin{bmatrix}\boldsymbol{\omega}\\0\end{bmatrix}\otimes\mathbf{q}^* \tag{59a}$$

$$\begin{bmatrix} A^T(\mathbf{q})\boldsymbol{\omega} \\ 0 \end{bmatrix} = \mathbf{q}^*\otimes\begin{bmatrix}\boldsymbol{\omega}\\0\end{bmatrix}\otimes\mathbf{q} \tag{59b}$$

$$Q_L(\mathbf{q})\Omega(\boldsymbol{\omega})Q_L(\mathbf{q}^*) = \Omega\left(A(\mathbf{q})\boldsymbol{\omega}\right) \tag{59c}$$

$$Q_L(\mathbf{q}^*)\Omega(\boldsymbol{\omega})Q_L(\mathbf{q}) = \Omega\left(A^T(\mathbf{q})\boldsymbol{\omega}\right) \tag{59d}$$

$$Q_R(\mathbf{q}^*)\Gamma(\boldsymbol{\omega})Q_R(\mathbf{q}) = \Gamma\left(A(\mathbf{q})\boldsymbol{\omega}\right) \tag{59e}$$

$$Q_R(\mathbf{q})\Gamma(\boldsymbol{\omega})Q_R(\mathbf{q}^*) = \Gamma\left(A^T(\mathbf{q})\boldsymbol{\omega}\right) \tag{59f}$$

Identities involving $Q_L(\mathbf{q}\otimes\bar{\mathbf{q}}^*)$ are given by

$$\Psi^T(\mathbf{q})Q_L(\mathbf{q}\otimes\bar{\mathbf{q}}^*)\Psi(\bar{\mathbf{q}}) = (\mathbf{q}^T\mathbf{q})(\bar{\mathbf{q}}^T\bar{\mathbf{q}})I_{3\times3} \tag{60a}$$

$$Q_L(\mathbf{q}\otimes\bar{\mathbf{q}}^*) = (\mathbf{q}^T\mathbf{q})^{-1}(\bar{\mathbf{q}}^T\bar{\mathbf{q}})^{-1}\Xi(\mathbf{q})A(\mathbf{q}\otimes\bar{\mathbf{q}}^*)\Xi^T(\bar{\mathbf{q}}) + \mathbf{q}\,\bar{\mathbf{q}}^T \tag{60b}$$

$$Q_L(\mathbf{q}\otimes\bar{\mathbf{q}}^*)\Xi(\bar{\mathbf{q}}) = (\mathbf{q}^T\mathbf{q})^{-1}\Xi(\mathbf{q})A(\mathbf{q}\otimes\bar{\mathbf{q}}^*) \tag{60c}$$

$$\Xi^T(\mathbf{q})Q_L(\mathbf{q}\otimes\bar{\mathbf{q}}^*) = (\bar{\mathbf{q}}^T\bar{\mathbf{q}})^{-1}A(\mathbf{q}\otimes\bar{\mathbf{q}}^*)\Xi^T(\bar{\mathbf{q}}) \tag{60d}$$

$$\Xi^T(\mathbf{q})Q_L(\mathbf{q}\otimes\bar{\mathbf{q}}^*)\Xi(\bar{\mathbf{q}}) = A(\mathbf{q}\otimes\bar{\mathbf{q}}^*) \tag{60e}$$

Identities involving $Q_R(\bar{\mathbf{q}}^*\otimes\mathbf{q})$ are given by

$$\Xi^T(\mathbf{q})Q_R(\bar{\mathbf{q}}^*\otimes\mathbf{q})\Xi(\bar{\mathbf{q}}) = (\mathbf{q}^T\mathbf{q})(\bar{\mathbf{q}}^T\bar{\mathbf{q}})I_{3\times3} \tag{61a}$$

$$Q_R(\bar{\mathbf{q}}^*\otimes\mathbf{q}) = (\mathbf{q}^T\mathbf{q})^{-1}(\bar{\mathbf{q}}^T\bar{\mathbf{q}})^{-1}\Psi(\mathbf{q})A(\mathbf{q}^*\otimes\bar{\mathbf{q}})\Psi^T(\bar{\mathbf{q}}) + \mathbf{q}\,\bar{\mathbf{q}}^T \tag{61b}$$

$$Q_R(\bar{\mathbf{q}}^*\otimes\mathbf{q})\Psi(\bar{\mathbf{q}}) = (\mathbf{q}^T\mathbf{q})^{-1}\Psi(\mathbf{q})A(\mathbf{q}^*\otimes\bar{\mathbf{q}}) \tag{61c}$$

$$\Psi^T(\mathbf{q})Q_R(\bar{\mathbf{q}}^*\otimes\mathbf{q}) = (\bar{\mathbf{q}}^T\bar{\mathbf{q}})^{-1}A(\mathbf{q}^*\otimes\bar{\mathbf{q}})\Psi^T(\bar{\mathbf{q}}) \tag{61d}$$

$$\Psi^T(\mathbf{q})Q_R(\bar{\mathbf{q}}^*\otimes\mathbf{q})\Psi(\bar{\mathbf{q}}) = A(\mathbf{q}^*\otimes\bar{\mathbf{q}}) \tag{61e}$$

Equation (51b) can be used to replace $Q_R(\bar{\mathbf{q}}^*\otimes\mathbf{q})$ with $Q_R(\mathbf{q}\otimes\bar{\mathbf{q}}^*)$ if needed.

Some interesting multiplication properties are given by

$$\begin{bmatrix}\boldsymbol{\omega}\\0\end{bmatrix}\otimes\mathbf{q} = \Omega(\boldsymbol{\omega})\mathbf{q} = \Xi(\mathbf{q})\boldsymbol{\omega} \tag{62a}$$

$$\mathbf{q}\otimes\begin{bmatrix}\boldsymbol{\omega}\\0\end{bmatrix} = \Gamma(\boldsymbol{\omega})\mathbf{q} = \Psi(\mathbf{q})\boldsymbol{\omega} \tag{62b}$$

The quaternion conjugate also obeys the same relationships:

$$\begin{bmatrix}\boldsymbol{\omega}\\0\end{bmatrix}\otimes\mathbf{q}^* = \Omega(\boldsymbol{\omega})\mathbf{q}^* = \Xi(\mathbf{q}^*)\boldsymbol{\omega} \tag{63a}$$

$$\mathbf{q}^*\otimes\begin{bmatrix}\boldsymbol{\omega}\\0\end{bmatrix} = \Gamma(\boldsymbol{\omega})\mathbf{q}^* = \Psi(\mathbf{q}^*)\boldsymbol{\omega} \tag{63b}$$

If we define $\boldsymbol{\omega}_{\mathcal{I}} \equiv A^T(\mathbf{q})\boldsymbol{\omega}$, which is used to denote the "space-referenced angular velocity" vector in kinematical equations, then the following relationships can be derived:

$$\mathbf{q}^*\otimes\begin{bmatrix}\boldsymbol{\omega}\\0\end{bmatrix} = (\mathbf{q}^T\mathbf{q})^{-1}\begin{bmatrix}\boldsymbol{\omega}_{\mathcal{I}}\\0\end{bmatrix}\otimes\mathbf{q}^* \tag{64a}$$

$$\Gamma(\boldsymbol{\omega})\mathbf{q}^* = (\mathbf{q}^T\mathbf{q})^{-1}\Omega(\boldsymbol{\omega}_{\mathcal{I}})\mathbf{q}^* \tag{64b}$$

$$\Psi(\mathbf{q}^*)\boldsymbol{\omega} = (\mathbf{q}^T\mathbf{q})^{-1}\Xi(\mathbf{q}^*)\boldsymbol{\omega}_{\mathcal{I}} \tag{64c}$$

and

$$\begin{bmatrix}\boldsymbol{\omega}\\0\end{bmatrix}\otimes\mathbf{q} = (\mathbf{q}^T\mathbf{q})^{-1}\mathbf{q}\otimes\begin{bmatrix}\boldsymbol{\omega}_{\mathcal{I}}\\0\end{bmatrix} \tag{65a}$$

$$\Omega(\boldsymbol{\omega})\mathbf{q} = (\mathbf{q}^T\mathbf{q})^{-1}\Gamma(\boldsymbol{\omega}_{\mathcal{I}})\mathbf{q} \tag{65b}$$

$$\Xi(\mathbf{q})\boldsymbol{\omega} = (\mathbf{q}^T\mathbf{q})^{-1}\Psi(\mathbf{q})\boldsymbol{\omega}_{\mathcal{I}} \tag{65c}$$

Note that the matrix $\Gamma(\boldsymbol{\omega})$ is *not* equivalent to $(\mathbf{q}^T\mathbf{q})^{-1}\Omega(\boldsymbol{\omega}_{\mathcal{I}})$, and $\Omega(\boldsymbol{\omega})$ is *not* equivalent to $(\mathbf{q}^T\mathbf{q})^{-1}\Gamma(\boldsymbol{\omega}_{\mathcal{I}})$ in general. As with Eq. (62), the relationships in Eqs. (63) and (64) are often used in estimation theory.[8] Also, if we define $\boldsymbol{\omega}_{\mathcal{B}} \equiv A(\mathbf{q})\boldsymbol{\omega}$, then the following relationships can be derived:

$$\begin{bmatrix} \boldsymbol{\omega} \\ 0 \end{bmatrix} \otimes \mathbf{q}^* = (\mathbf{q}^T\mathbf{q})^{-1}\mathbf{q}^* \otimes \begin{bmatrix} \boldsymbol{\omega}_{\mathcal{B}} \\ 0 \end{bmatrix} \tag{66a}$$

$$\Omega(\boldsymbol{\omega})\mathbf{q}^* = (\mathbf{q}^T\mathbf{q})^{-1}\Gamma(\boldsymbol{\omega}_{\mathcal{B}})\mathbf{q}^* \tag{66b}$$

$$\Xi(\mathbf{q}^*)\boldsymbol{\omega} = (\mathbf{q}^T\mathbf{q})^{-1}\Psi(\mathbf{q}^*)\boldsymbol{\omega}_{\mathcal{B}} \tag{66c}$$

and

$$\mathbf{q} \otimes \begin{bmatrix} \boldsymbol{\omega} \\ 0 \end{bmatrix} = (\mathbf{q}^T\mathbf{q})^{-1} \begin{bmatrix} \boldsymbol{\omega}_{\mathcal{B}} \\ 0 \end{bmatrix} \otimes \mathbf{q} \tag{67a}$$

$$\Gamma(\boldsymbol{\omega})\mathbf{q} = (\mathbf{q}^T\mathbf{q})^{-1}\Omega(\boldsymbol{\omega}_{\mathcal{B}})\mathbf{q} \tag{67b}$$

$$\Psi(\mathbf{q})\boldsymbol{\omega} = (\mathbf{q}^T\mathbf{q})^{-1}\Xi(\mathbf{q})\boldsymbol{\omega}_{\mathcal{B}} \tag{67c}$$

Note that the matrix $\Omega(\boldsymbol{\omega})$ is *not* equivalent to $(\mathbf{q}^T\mathbf{q})^{-1}\Gamma(\boldsymbol{\omega}_{\mathcal{B}})$, and $\Gamma(\boldsymbol{\omega})$ is *not* equivalent to $(\mathbf{q}^T\mathbf{q})^{-1}\Omega(\boldsymbol{\omega}_{\mathcal{B}})$ in general. We also should note that $\boldsymbol{\omega}_{\mathcal{B}}$ is not referred to as the "body-referenced angular velocity" since $\boldsymbol{\omega}$ is most often already given in body coordinates. The vector $\boldsymbol{\omega}_{\mathcal{B}}$ is merely another rotated vector from $\boldsymbol{\omega}$.

MORE KINEMATICS

In this section, we will show more kinematically relationships used for estimation and control purposes. The first involves the kinematical relationship for $\boldsymbol{\delta}\mathbf{q}$. Unless otherwise stated, all quaternions used in this section are assumed to be normalized. Taking the time derivative of Eq. (52) gives

$$\frac{d}{dt}\boldsymbol{\delta}\mathbf{q} = \frac{d}{dt}[\mathbf{q}] \otimes \bar{\mathbf{q}}^{-1} + \mathbf{q} \otimes \frac{d}{dt}\left[\bar{\mathbf{q}}^{-1}\right] \tag{68}$$

We now need to determine an expression for the derivative of $\bar{\mathbf{q}}^{-1}$. The estimated/desired quaternion kinematics model follows

$$\frac{d}{dt}\bar{\mathbf{q}} = \frac{1}{2}\Xi(\bar{\mathbf{q}})\bar{\boldsymbol{\omega}} = \frac{1}{2}\Omega(\bar{\boldsymbol{\omega}})\bar{\mathbf{q}} \tag{69}$$

Taking the time derivative of $\bar{\mathbf{q}} \otimes \bar{\mathbf{q}}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$ gives

$$\frac{d}{dt}[\bar{\mathbf{q}}] \otimes \bar{\mathbf{q}}^{-1} + \bar{\mathbf{q}} \otimes \frac{d}{dt}\left[\bar{\mathbf{q}}^{-1}\right] = \mathbf{0} \tag{70}$$

Substituting Eq. (69) into Eq. (70) gives

$$\frac{1}{2}\Omega(\bar{\boldsymbol{\omega}})\bar{\mathbf{q}} \otimes \bar{\mathbf{q}}^{-1} + \bar{\mathbf{q}} \otimes \frac{d}{dt}\left[\bar{\mathbf{q}}^{-1}\right] = \mathbf{0} \tag{71}$$

Since $\bar{\mathbf{q}} \otimes \bar{\mathbf{q}}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$, and using the identity in Eq. (62a), then Eq. (71) reduces down to

$$\frac{1}{2}\begin{bmatrix} \bar{\boldsymbol{\omega}} \\ 0 \end{bmatrix} + \bar{\mathbf{q}} \otimes \frac{d}{dt}\left[\bar{\mathbf{q}}^{-1}\right] = \mathbf{0} \tag{72}$$

Solving Eq. (72) for the derivative of $\bar{\mathbf{q}}^{-1}$ yields

$$\frac{d}{dt}\left[\bar{\mathbf{q}}^{-1}\right] = -\frac{1}{2}\bar{\mathbf{q}}^{-1} \otimes \begin{bmatrix} \bar{\boldsymbol{\omega}} \\ 0 \end{bmatrix} = -\frac{1}{2}\Gamma(\bar{\boldsymbol{\omega}})\bar{\mathbf{q}}^{-1} = -\frac{1}{2}\Psi(\bar{\mathbf{q}}^{-1})\bar{\boldsymbol{\omega}} \tag{73}$$

where the identities in Eq. (63b) are used in Eq. (73).

Substituting Eqs. (73) and (11) into Eq. (68), and using the definition of the error quaternion in Eq. (52) gives

$$\frac{d}{dt}\boldsymbol{\delta}\mathbf{q} = \frac{1}{2}\left\{\begin{bmatrix} \boldsymbol{\omega} \\ 0 \end{bmatrix} \otimes \boldsymbol{\delta}\mathbf{q} - \boldsymbol{\delta}\mathbf{q} \otimes \begin{bmatrix} \bar{\boldsymbol{\omega}} \\ 0 \end{bmatrix}\right\} \tag{74}$$

We now define the following error angular velocity: $\boldsymbol{\delta\omega} \equiv \boldsymbol{\omega} - \bar{\boldsymbol{\omega}}$. Substituting $\boldsymbol{\omega} = \bar{\boldsymbol{\omega}} + \boldsymbol{\delta\omega}$ into Eq. (74) leads to

$$\frac{d}{dt}\boldsymbol{\delta}\mathbf{q} = \frac{1}{2}\left\{\begin{bmatrix} \bar{\boldsymbol{\omega}} \\ 0 \end{bmatrix} \otimes \boldsymbol{\delta}\mathbf{q} - \boldsymbol{\delta}\mathbf{q} \otimes \begin{bmatrix} \bar{\boldsymbol{\omega}} \\ 0 \end{bmatrix}\right\} + \frac{1}{2}\begin{bmatrix} \boldsymbol{\delta\omega} \\ 0 \end{bmatrix} \otimes \boldsymbol{\delta}\mathbf{q} \tag{75}$$

Next, using the helpful identities in Eq. (62), replacing $\mathbf{q}$ with $\boldsymbol{\delta}\mathbf{q}$, leads to

$$\frac{d}{dt}\boldsymbol{\delta}\mathbf{q} = -\begin{bmatrix} [\bar{\boldsymbol{\omega}}\times]\boldsymbol{\delta\varrho} \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \boldsymbol{\delta\omega} \\ 0 \end{bmatrix} \otimes \boldsymbol{\delta}\mathbf{q} \tag{76}$$

Equation (76) can be written as

$$\frac{d}{dt}\boldsymbol{\delta\varrho} = -[\bar{\boldsymbol{\omega}}\times]\boldsymbol{\delta\varrho} + \frac{1}{2}\left(\delta q_4\boldsymbol{\delta\omega} - [\boldsymbol{\delta\omega}\times]\boldsymbol{\delta\varrho}\right) \tag{77a}$$

$$\frac{d}{dt}\delta q_4 = -\boldsymbol{\delta\omega}^T\boldsymbol{\delta\varrho} \tag{77b}$$

Equation (77a) can also be derived by taking the derivative of Eq. (53a), which leads to

$$\frac{d}{dt}\boldsymbol{\delta\varrho} = \frac{1}{2}\left[\Xi^T(\bar{\mathbf{q}})\Xi(\mathbf{q})\boldsymbol{\omega} - \Xi^T(\mathbf{q})\Xi(\bar{\mathbf{q}})\bar{\boldsymbol{\omega}}\right] \tag{78}$$

Equation (57a) can used to prove that Eq. (77a) is equivalent to Eq. (78). This identity is valid for non-normalized quaternions as well.

Another useful kinematical equation is the derivative of $\boldsymbol{\delta}\mathbf{q}_{\mathcal{I}}$, which follows

$$\frac{d}{dt}\boldsymbol{\delta}\mathbf{q}_{\mathcal{I}} = \begin{bmatrix} [\bar{\boldsymbol{\omega}}_{\mathcal{I}}\times]\boldsymbol{\delta\varrho}_{\mathcal{I}} \\ 0 \end{bmatrix} + \frac{1}{2}\boldsymbol{\delta}\mathbf{q}_{\mathcal{I}} \otimes \begin{bmatrix} \boldsymbol{\delta\omega}_{\mathcal{I}} \\ 0 \end{bmatrix} \tag{79}$$

where $\bar{\boldsymbol{\omega}}_{\mathcal{I}} \equiv A^T(\bar{\mathbf{q}})\bar{\boldsymbol{\omega}}$ and $\boldsymbol{\delta\omega}_{\mathcal{I}} \equiv \boldsymbol{\omega}_{\mathcal{I}} - \bar{\boldsymbol{\omega}}_{\mathcal{I}}$. Equation (79) can be written as

$$\frac{d}{dt}\boldsymbol{\delta\varrho}_{\mathcal{I}} = [\bar{\boldsymbol{\omega}}_{\mathcal{I}}\times]\boldsymbol{\delta\varrho}_{\mathcal{I}} + \frac{1}{2}\left(\delta q_4\boldsymbol{\delta\omega}_{\mathcal{I}} + [\boldsymbol{\delta\omega}_{\mathcal{I}}\times]\boldsymbol{\delta\varrho}_{\mathcal{I}}\right) \tag{80a}$$

$$\frac{d}{dt}\delta q_4 = -\boldsymbol{\delta\omega}_{\mathcal{I}}^T\boldsymbol{\delta\varrho}_{\mathcal{I}} \tag{80b}$$

Equation (80a) can also be derived by taking the derivative of Eq. (55), which leads to

$$\frac{d}{dt}\boldsymbol{\delta\varrho}_{\mathcal{I}} = \frac{1}{2}\left[\Psi^T(\bar{\mathbf{q}})\Psi(\mathbf{q})\boldsymbol{\omega}_{\mathcal{I}} - \Psi^T(\mathbf{q})\Psi(\bar{\mathbf{q}})\bar{\boldsymbol{\omega}}_{\mathcal{I}}\right] \tag{81}$$

Equation (57c) can used to prove that Eqn. (80a) is equivalent to Eq. (81). This identity is valid for non-normalized quaternions as well. By comparing Eq. (77b) with (80b) we see that $\boldsymbol{\delta\omega}^T\boldsymbol{\delta\varrho} = \boldsymbol{\delta\omega}_{\mathcal{I}}^T\boldsymbol{\delta\varrho}_{\mathcal{I}}$, which is true only if the quaternions are normalized. If this is not true, then the following identity can be used:

$$\boldsymbol{\delta\omega}^T\boldsymbol{\delta\varrho} = \left[(\mathbf{q}^T\mathbf{q})^{-1}A^T(\mathbf{q})\boldsymbol{\omega} - (\bar{\mathbf{q}}^T\bar{\mathbf{q}})^{-1}A^T(\bar{\mathbf{q}})\bar{\boldsymbol{\omega}}\right]^T\boldsymbol{\delta\varrho}_{\mathcal{I}} \tag{82}$$

The quaternion kinematics equation in Eq. (11) can also be written as

$$\frac{d}{dt}\mathbf{q} = \frac{1}{2}\Psi(\mathbf{q})\boldsymbol{\omega}_{\mathcal{I}} = \frac{1}{2}\Gamma(\boldsymbol{\omega}_{\mathcal{I}})\mathbf{q} \tag{83a}$$

$$\boldsymbol{\omega}_{\mathcal{I}} = 2\,\Psi^T(\mathbf{q})\,\dot{\mathbf{q}} \tag{83b}$$

where the definition of the attitude matrix in Eq. (9) can be used to easily prove Eq. (83a). The derivative of the matrix $\Xi(\mathbf{q})$ is given by[1]

$$\frac{d}{dt}\Xi(\mathbf{q}) = \frac{1}{2}\Gamma(\boldsymbol{\omega}_{\mathcal{I}})\Xi(\mathbf{q}) \tag{84a}$$

$$= \frac{1}{2}\Omega(\boldsymbol{\omega})\Xi(\mathbf{q}) + \Xi(\mathbf{q})[\boldsymbol{\omega}\times] \tag{84b}$$

$$= -\frac{1}{2}\mathbf{q}\,\boldsymbol{\omega}^T + \frac{1}{2}\Xi(\mathbf{q})[\boldsymbol{\omega}\times] \tag{84c}$$

The derivative of the matrix $\Psi(\mathbf{q})$ is given by

$$\frac{d}{dt}\Psi(\mathbf{q}) = \frac{1}{2}\Omega(\boldsymbol{\omega})\Psi(\mathbf{q}) \tag{85a}$$

$$= \frac{1}{2}\Gamma(\boldsymbol{\omega}_{\mathcal{I}})\Psi(\mathbf{q}) - \Psi(\mathbf{q})[\boldsymbol{\omega}_{\mathcal{I}}\times] \tag{85b}$$

$$= -\frac{1}{2}\mathbf{q}\,\boldsymbol{\omega}_{\mathcal{I}}^T - \frac{1}{2}\Psi(\mathbf{q})[\boldsymbol{\omega}_{\mathcal{I}}\times] \tag{85c}$$

Equations (84) and (85) lead to some more identities that are valid for non-normalized quaternions:

$$(\mathbf{q}^T\mathbf{q})^{-1}\Gamma(\boldsymbol{\omega}_{\mathcal{I}})\Xi(\mathbf{q}) = \Omega(\boldsymbol{\omega})\Xi(\mathbf{q}) + 2\,\Xi(\mathbf{q})[\boldsymbol{\omega}\times] \tag{86a}$$

$$= -\mathbf{q}\,\boldsymbol{\omega}^T + \Xi(\mathbf{q})[\boldsymbol{\omega}\times] \tag{86b}$$

and

$$\Omega(\boldsymbol{\omega})\Psi(\mathbf{q}) = (\mathbf{q}^T\mathbf{q})^{-1}\left\{\Gamma(\boldsymbol{\omega}_{\mathcal{I}})\Psi(\mathbf{q}) - 2\,\Psi(\mathbf{q})[\boldsymbol{\omega}_{\mathcal{I}}\times]\right\} \tag{87a}$$

$$= -(\mathbf{q}^T\mathbf{q})^{-1}\left\{\mathbf{q}\,\boldsymbol{\omega}_{\mathcal{I}}^T + \Psi(\mathbf{q})[\boldsymbol{\omega}_{\mathcal{I}}\times]\right\} \tag{87b}$$

Note the relation of the identities in Eq. (87) to Eq. (29a). Derivatives of the matrices $Q_L(\mathbf{q})$ and $Q_R(\mathbf{q})$ are given by[1]

$$\frac{d}{dt}Q_L(\mathbf{q}) = \frac{1}{2}\Omega(\boldsymbol{\omega})Q_L(\mathbf{q}) = \frac{1}{2}Q_L(\mathbf{q})\Omega(\boldsymbol{\omega}_{\mathcal{I}}) \tag{88a}$$

$$\frac{d}{dt}Q_R(\mathbf{q}) = \frac{1}{2}Q_R(\mathbf{q})\Gamma(\boldsymbol{\omega}) = \frac{1}{2}\Gamma(\boldsymbol{\omega}_{\mathcal{I}})Q_R(\mathbf{q}) \tag{88b}$$

where the identities in Eqs. (59d) and (59f) have been used in Eq. (88).

ACKNOWLEDGEMENT

Many of these identities listed here have been taken from Dr. Malcolm Shuster's survey paper. The author wishes to thank Dr. Shuster and Dr. F. Landis Markley for the inspiration throughout the years to seek the others that are listed in this document. Also, the author wishes to thank Dr. Mark E. Pittelkau and Dr. Daniel Choukroun for many helpful discussions and inputs to other identities that appear here.

REFERENCES

1. M. D. Shuster, "A Survey of Attitude Representations," *Journal of the Astronautical Sciences*, Vol. 41, No. 4, Oct.-Dec. 1993, pp. 439–517.

2. W. R. Hamilton, *Elements of Quaternions*, Longmans, Green and Co., London, England, 1866.

3. H. Schaub and J. L. Junkins, *Analytical Mechanics of Aerospace Systems*, American Institute of Aeronautics and Astronautics, Inc., New York, NY, 2003.

4. M. D. Shuster, "Attitude Determination from Vector Observations," *Journal of Guidance and Control*, Vol. 4, No. 1, Jan.-Feb. 1981, pp. 70–77.

5. M. E. Pittelkau, "Square Root Quaternion Estimation," *AIAA/AAS Astrodynamics Specialist Conference and Exhibit*, Monterey, CA, Aug. 2002, AIAA-2002-4914.

6. D. Choukroun, I. Y. Bar-Itzhack, and Y. Oshman, "Novel Quaternion Kalman Filter," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 42, No. 1, Jan. 2006, pp. 174–190.

7. F. L. Markley, "Attitude Estimation or Quaternion Estimation," *The John L. Junkins Astrodynamics Symposium*, College Station, TX, May 2003, AAS-03-264.

8. E. J. Lefferts, F. L. Markley, and M. D. Shuster, "Kalman Filtering for Spacecraft Attitude Estimation," *Journal of Guidance, Control, and Dynamics*, Vol. 5, No. 5, Sept.-Oct. 1982, pp. 417–429.

9. L. Fallon, "Quaternions," *Spacecraft Attitude Determination and Control*, edited by J. R. Wertz, Kluwer Academic Publishers, The Netherlands, 1978, Appendix D.

10. J. L. Crassidis and J. L. Junkins, *Optimal Estimation of Dynamic Systems*, Chapman & Hall/CRC, Boca Raton, FL, 2004.

11. R. A. Paielli and R. E. Bach, "Attitude Control with Realization of Linear Error Dynamics," *Journal of Guidance, Control and Dynamics*, Vol. 16, No. 1, Jan.-Feb. 1993, pp. 182–189.